INTEGER MULTIPLICATION VIA FLOATING POINT

ABSTRACT. It is no secret that the floating point units on modern processors outshine the integer units by a significant margin. However, hardware floating point arithmetic is either subject to roundoff error or a reduced range of values where the roundoff can be avoided. With support for the fused multiply-add operation being ubiquitous, we have sufficient precision for competitive multiprecision integer arithmetic, and its performance is compared with the gnu multiprecision library (gmp), which is highly optimized but uses the integer units exclusively.

1. NOTATION

$\mathbb{e}$	$2.71\ldots$
π	$3.14\ldots$
$\mathbb{i}$	$\sqrt{-1}$ or something whose square is -1 in the base ring.
$\mathrm{EvalPoly}(a,\vec{x})$	$\sum_{0\le i<n} x_i a^i$, $\vec{x}=(x_0,\ldots,x_{n-1})$
ω_b^a	$\mathbb{e}^{2a\pi\mathbb{i}/b}$ when the base ring is $\mathbb{C}$, otherwise some compatible b^{th} root of unity.
$\overline{i}^k$	length k bit-reversal of i. Only defined for $0 \le i < 2^k$. $\overline{12}^4 = \overline{1100_2}^4 = 0011_2 = 3$

2. FFT/IFFT

2.1. **convolutions.** For any two vectors $\vec{x}$ and $\vec{y}$ of length n, the *cyclic convolution* $\vec{x} * \vec{y}$ is the length n vector formed by the product of the corresponding polynomials modulo $\phi^n - 1$:

$$\mathrm{EvalPoly}(\phi, \vec{x} * \vec{y}) := \mathrm{EvalPoly}(\phi, \vec{x}) \cdot \mathrm{EvalPoly}(\phi, \vec{y}) \bmod \phi^n - 1.$$

Since the coefficients $\vec{x}*\vec{y}$ appear linearly on the left hand side, they can be computed by an evaluation (fft) and interpolation (ifft) scheme at the n roots of $\phi^n - 1$. A more general convolution can be defined by

$$\mathrm{EvalPoly}(\phi, \vec{x} *^a \vec{y}) := \mathrm{EvalPoly}(\phi, \vec{x}) \cdot \mathrm{EvalPoly}(\phi, \vec{y}) \bmod \phi^n - a^n.$$

Replacing ϕ by $a\phi$ has the effect of turning this into an ordinary cyclic convolution after rescaling the input and output by powers of a. Hence, it can be computed the same way. When $a^n = -1$ this is called a *negacyclic convolution*, and when $a^n = \pm\mathbb{i}$ this is called a *right angle convolution*.

The applications of convolutions to polynomial multiplication should be clear: as long as the degree of the product is less than the transformation length n, the product of the polynomials can be computed by the cyclic convolution of the coefficients of the input polynomials zero padded to length n. For integer multiplication, we usually view the integers to be multiplied as elements of $\mathbb{Z}[x]$ evaluated at some power of two $x = 2^k$. In this way, integer multiplication can be derived from multiplication in $\mathbb{Z}[x]$ where we have the extra freedom of being able to choose the size of the coefficients of the polynomials in $\mathbb{Z}[x]$. The convolution itself can be performed over any ring with n-th roots of unity, including the following.

- $\mathbb{C}$: Some kind of approximate arithmetic is needed as the roots of unity are irrational.
- $\mathbb{Z}/(2^m+1)\mathbb{Z}$: Here 2 is a $2m$-th root of unity and the corresponding convolution was originally used by Schönhage and Strassen to multiply large integers.
- $\mathbb{F}_p$: Here we only require that n divides $p-1$ for some prime p for the existence of the n-th roots of unity.
- $\mathbb{Z}/p_1p_2\cdots p_m\mathbb{Z}$: The convolution over this ring can be obtained via the Chinese remainder theorem and the convolutions over each of $\mathbb{F}_{p_1}$, $\mathbb{F}_{p_2}$, $\ldots$, $\mathbb{F}_{p_m}$.

2.2. **bit reversals and vector lengths of the form 2^k.** Unless the output of the fft is specifically needed to be *in order*, that is, in the usual order

$$\left\{\text{EvalPoly}(\omega_{2^k}^{i}, \vec{x})\right\}_{0 \le i < 2^k},$$

there is no reason not to give the output in bit-reversed order

$$\text{fft}(\vec{x}) := \left\{\text{EvalPoly}(\omega_{2^k}^{\bar{i}^k}, \vec{x})\right\}_{0 \le i < 2^k}.$$

The reason is that the bit-reversed output is much simpler and faster (compare the performance of fftw3 in Figure 10) because it groups similar outputs close together. For example, $\text{EvalPoly}(1, \vec{x})$ and $\text{EvalPoly}(-1, \vec{x})$ are very similar computationally and are right next to each other in the bit-reversed output, but are very far apart in the usual order. Therefore, we will restrict exclusively to bit-reversed outputs.

The usual sequence [2] for calculating a length 16 fft follows the columns below and ends with the fft in the last column. Notice that for a input sequence of length 2^k, we make k passes over the data, which gives the whole algorithm a run time complexity of $O(k2^k)$ assuming arithmetic operations in the ring are constant time.

(1)

x_i	y_i	z_i	w_i	$\text{fft}(\vec{x})$
x_0	$x_0 + x_8$	$y_0 + y_4$	$z_0 + z_2$	$w_0 + w_1$
x_1	$x_1 + x_9$	$y_1 + y_5$	$z_1 + z_3$	$(w_0 - w_1)$
x_2	$x_2 + x_{10}$	$y_2 + y_6$	$(z_0 - z_2)\omega_4^0$	$w_2 + w_3$
x_3	$x_3 + x_{11}$	$y_3 + y_7$	$(z_2 - z_3)\omega_4^1$	$(w_2 - w_3)$
x_4	$x_4 + x_{12}$	$(y_0 - y_4)\omega_8^0$	$z_4 + z_6$	$w_4 + w_5$
x_5	$x_5 + x_{13}$	$(y_1 - y_5)\omega_8^1$	$z_5 + z_7$	$(w_4 - w_5)$
x_6	$x_6 + x_{14}$	$(y_2 - y_6)\omega_8^2$	$(z_4 - z_6)\omega_4^0$	$w_6 + w_7$
x_7	$x_7 + x_{15}$	$(y_3 - y_7)\omega_8^3$	$(z_5 - z_7)\omega_4^1$	$(w_6 - w_7)$
x_8	$(x_0 - x_8)\omega_{16}^0$	$y_8 + y_{12}$	$z_8 + z_{10}$	$w_8 + w_9$
x_9	$(x_1 - x_9)\omega_{16}^1$	$y_9 + y_{13}$	$z_9 + z_{11}$	$(w_8 - w_9)$
x_{10}	$(x_2 - x_{10})\omega_{16}^2$	$y_{10} + y_{14}$	$(z_8 - z_{10})\omega_4^0$	$w_{10} + w_{11}$
x_{11}	$(x_3 - x_{11})\omega_{16}^3$	$y_{11} + y_{15}$	$(z_9 - z_{11})\omega_4^1$	$(w_{10} - w_{11})$
x_{12}	$(x_4 - x_{12})\omega_{16}^4$	$(y_8 - y_{12})\omega_8^0$	$z_{12} + z_{14}$	$w_{12} + w_{13}$
x_{13}	$(x_5 - x_{13})\omega_{16}^5$	$(y_9 - y_{13})\omega_8^1$	$z_{13} + z_{15}$	$(w_{12} - w_{13})$
x_{14}	$(x_6 - x_{14})\omega_{16}^6$	$(y_{10} - y_{14})\omega_8^2$	$(z_{12} - z_{14})\omega_4^0$	$w_{14} + w_{15}$
x_{15}	$(x_7 - x_{15})\omega_{16}^7$	$(y_{11} - y_{15})\omega_8^3$	$(z_{13} - z_{15})\omega_4^1$	$(w_{14} - w_{15})$

One problem with this approach is that each column accesses many different *twiddle factors* ω_b^a. In the case of a Schönhage–Strassen fft where the base ring is $\mathbb{Z}/(2^m + 1)\mathbb{Z}$, this doesn't matter because each twiddle factor is a power of two and implemented via bit shifts. In other cases, these twiddle factors have to be either computed on the fly or precomputed and then retrieved from storage. In the case of precomputation, we have to have in memory the table

$$1, \omega_2^1, 1, \omega_4^1, 1, \omega_8^1, \omega_8^2, \omega_8^3, 1, \omega_{16}^1, \omega_{16}^2, \omega_{16}^3, \omega_{16}^4, \omega_{16}^5, \omega_{16}^6, \omega_{16}^7, 1, \dots \tag{2}$$

so that the columns can access this table sequentially. However, such a table is nice because once it is extended to accommodate an fft of a certain length, it can be reused for all ffts of smaller length.

By rearranging the twiddle factors as $(\omega = \omega_{16})$

(3)

x_i	y_i	z_i	w_i	$\text{fft}(\vec{x})$
x_0	$x_0 + \omega^0 x_8$	$y_0 + \omega^0 y_4$	$z_0 + \omega^0 z_2$	$w_0 + \omega^0 w_1$
x_1	$x_1 + \omega^0 x_9$	$y_1 + \omega^0 y_5$	$z_1 + \omega^0 z_3$	$w_0 + \omega^8 w_1$
x_2	$x_2 + \omega^0 x_{10}$	$y_2 + \omega^0 y_6$	$z_0 + \omega^8 z_2$	$w_2 + \omega^4 w_3$
x_3	$x_3 + \omega^0 x_{11}$	$y_3 + \omega^0 y_7$	$z_1 + \omega^8 z_3$	$w_2 + \omega^{12} w_3$
x_4	$x_4 + \omega^0 x_{12}$	$y_0 + \omega^8 y_4$	$z_4 + \omega^4 z_6$	$w_4 + \omega^2 w_5$
x_5	$x_5 + \omega^0 x_{13}$	$y_1 + \omega^8 y_5$	$z_5 + \omega^4 z_7$	$w_4 + \omega^{10} w_5$
x_6	$x_6 + \omega^0 x_{14}$	$y_2 + \omega^8 y_6$	$z_4 + \omega^{12} z_6$	$w_6 + \omega^6 w_7$
x_7	$x_7 + \omega^0 x_{15}$	$y_3 + \omega^8 y_7$	$z_5 + \omega^{12} z_7$	$w_6 + \omega^{14} w_7$
x_8	$x_0 + \omega^8 x_8$	$y_8 + \omega^4 y_{12}$	$z_8 + \omega^2 z_{10}$	$w_8 + \omega^1 w_9$
x_9	$x_1 + \omega^8 x_9$	$y_9 + \omega^4 y_{13}$	$z_9 + \omega^2 z_{11}$	$w_8 + \omega^9 w_9$
x_{10}	$x_2 + \omega^8 x_{10}$	$y_{10} + \omega^4 y_{14}$	$z_8 + \omega^{10} z_{10}$	$w_{10} + \omega^5 w_{11}$
x_{11}	$x_3 + \omega^8 x_{11}$	$y_{11} + \omega^4 y_{15}$	$z_9 + \omega^{10} z_{11}$	$w_{10} + \omega^{13} w_{11}$
x_{12}	$x_4 + \omega^8 x_{12}$	$y_8 + \omega^{12} y_{12}$	$z_{12} + \omega^6 z_{14}$	$w_{12} + \omega^3 w_{13}$
x_{13}	$x_5 + \omega^8 x_{13}$	$y_9 + \omega^{12} y_{13}$	$z_{13} + \omega^6 z_{15}$	$w_{12} + \omega^{11} w_{13}$
x_{14}	$x_6 + \omega^8 x_{14}$	$y_{10} + \omega^{12} y_{14}$	$z_{12} + \omega^{14} z_{14}$	$w_{14} + \omega^7 w_{15}$
x_{15}	$x_7 + \omega^8 x_{15}$	$y_{11} + \omega^{12} y_{15}$	$z_{13} + \omega^{14} z_{15}$	$w_{14} + \omega^{15} w_{15}$

there are still the same number of twiddle factor multiplications [1], but each twiddle factor itself can be reused in each column. Also, as with the previous method, there is a universal (bit-reversed) table

$$1,\quad \omega_2^1,\quad \omega_4^1, \omega_4^3,\quad \omega_8^1, \omega_8^5, \omega_8^3, \omega_8^7,\quad \omega_{16}^1, \omega_{16}^9, \omega_{16}^5, \omega_{16}^{13}, \omega_{16}^3, \omega_{16}^{11}, \omega_{16}^7, \omega_{16}^{15},\quad \ldots \tag{4}$$

that can be reused. The difference here is that the portion that is used for a specific fft is now half the size (making note of $\omega_4^3 = -\omega_4^1$, $\omega_8^5 = -\omega_8^1$, $\omega_8^7 = -\omega_8^3$, etc.).

Since the output of the fft is given in bit-reversed order, the inverse operation cannot simply replace ω by ω^{-1} and use the same calculation sequence. Thus the ifft has to invert the left-to-right operation by working from the right to the left and inverting each basic operation. This gives slightly different data access patterns but involves essentially the same calculations. For example, the operation $w_8 = z_8 + \omega^2 z_{10}, w_{10} = z_8 - \omega^2 z_{10}$ is inverted by $2z_8 = w_8 + w_{10}, 2z_{10} = \omega^{-2}(w_8 - w_{10})$ and the negative power $\omega^{-\bar{i}^k}$ can be looked up in the table by flipping all bits of i except the highest.

2.3. fft on sizes other than powers of two.

Suppose the vector to transform has length thrice a power of two: $x_0, x_1, \ldots, x_{3\cdot 2^k - 1}$. If, for each $0 \le i < 2^k$, we first perform

$$\begin{aligned} y_i^{(0)} &= x_{i+0\cdot 2^k} + x_{i+1\cdot 2^k} + x_{i+2\cdot 2^k} \\ y_i^{(1)} &= \omega_{3\cdot 2^k}^{i}(x_{i+0\cdot 2^k} + \omega_3^1 x_{i+1\cdot 2^k} + \omega_3^{-1} x_{i+2\cdot 2^k}) \\ y_i^{(2)} &= \omega_{3\cdot 2^k}^{-i}(x_{i+0\cdot 2^k} + \omega_3^{-1} x_{i+1\cdot 2^k} + \omega_3^1 x_{i+2\cdot 2^k}), \end{aligned}$$

then the three ffts of the length 2^k sequences $y^{(0)}, y^{(1)}, y^{(2)}$ will give the fft of the original sequence. However, while the transforms of $y^{(0)}$ and $y^{(1)}$ will have the same ordering, the sequence $y^{(2)}$ will transform to a different ordering due to the negative power. More precisely,

$$\begin{aligned} \text{EvalPoly}(\omega_{3\cdot 2^k}^{3i}, x) &= \text{EvalPoly}(\omega_{2^k}^{i}, y^{(0)}) \\ \text{EvalPoly}(\omega_{3\cdot 2^k}^{3i+1}, x) &= \text{EvalPoly}(\omega_{2^k}^{i}, y^{(1)}) \\ \text{EvalPoly}(\omega_{3\cdot 2^k}^{3i-1}, x) &= \text{EvalPoly}(\omega_{2^k}^{i}, y^{(2)}). \end{aligned}$$

[1] The difference between these two calculation sequences corresponds to what is loosely called a *decimation in time* and a *decimation in frequency*, though it is not clear which is which.

Table 1. Integer multiplication with complex 53 bit ffts: The two input numbers are viewed as polynomials evaluated at 2^m so that the fft has inputs in the range $[0, 2^m)$. Rounding the output coefficients from the ifft to the nearest integer is eventually unreliable due to intermediate rounding errors: let $f(m)$ denote the maximum bit size of an answer which can be guaranteed correct by this algorithm.

m	upper bound on $f(m)$
22	3008
21	11968
20	30464
19	117632
18	417280
17	1246592
16	1984000

2.4. **considerations for the base ring $\mathbb{F}_p$.** In order to support multiplication in this ring in floating point for decently sized p (say 50 bits), it is necessary to have hardware support for a fused multiply-add operation, since the 100 bit products *must* be represented exactly as the sum of two floating point numbers.

2.5. **considerations for the base ring $\mathbb{C}$.** When arithmetic in the base ring necessarily includes roundoff error, it is important for the fft to be implemented as accurately as possible. Since both (1) and (3) compute the bit-reversed fft, we are free to use either. Using (3) for the forward transform and inverting (1) for the inverse transform gives calculation sequences for both that consist entirely of fused-multiply-add operations. The ifft will read often from the table (2), but this is of little consequence since large convolution lengths are not feasible for this base ring (see Table 1).

The case of real input and output data is important and must be optimized in order to be competitive. Feeding in purely real data to the fft and getting back real data from the ifft represents only a 50% utilization of the circuits (1) or (3), since the data possesses certain symmetries under complex conjugation in each column of the calculation. First, and most elementary, two real data sets a and b of the same length can be transformed with one complex transformation of the same length, and this is nothing but the linearity of the fft operation:

$$\begin{aligned} \operatorname{EvalPoly}(\omega, \vec{a} + \mathbb{i}\vec{b}) &= \operatorname{EvalPoly}(\omega, \vec{a}) + \mathbb{i}\operatorname{EvalPoly}(\omega, \vec{b}) \\ \overline{\operatorname{EvalPoly}(\omega^{-1}, \vec{a} + \mathbb{i}\vec{b})} &= \operatorname{EvalPoly}(\omega, \vec{a}) - \mathbb{i}\operatorname{EvalPoly}(\omega, \vec{b}) \end{aligned}$$

This is fine for doubling the processing speed of an even number of real data sets of the same length, but if we would like to process one real data set twice as fast, the following identity can be used.

$$\operatorname{EvalPoly}(\omega, (x_0, \dots, x_{2n-1})) = \operatorname{EvalPoly}(\omega^2, (x_0, x_2, \dots, x_{2n-2})) + \omega \operatorname{EvalPoly}(\omega^2, (x_1, x_3, \dots, x_{2n-1})).$$

The cyclic convolution of real coefficients $\{a_i\}$ and $\{b_i\}$ to form product coefficients $\{c_i\}$ can then take the form

$$\begin{array}{c} a_0 + \mathbb{i}b_0 \\ a_1 + \mathbb{i}b_1 \\ \dots \\ a_{15} + \mathbb{i}b_{15} \end{array} \underset{\text{fft}_{16}}{\Longrightarrow} \begin{array}{c} f_0 \\ f_8 \\ f_4 \\ f_{12} \\ \dots \\ f_{14} \\ f_1 \\ \dots \\ f_7 \\ f_{15} \end{array} \Longrightarrow \begin{array}{r} \hat{a}_0 \cdot \hat{b}_0 = \hat{c}_0 \\ \hat{a}_8 \cdot \hat{b}_8 = \hat{c}_8 \\ \hat{a}_4 \cdot \hat{b}_4 = \hat{c}_4 \\ \dots \\ \hat{a}_{14} \cdot \hat{b}_{14} = \hat{c}_{14} \\ \hat{a}_1 \cdot \hat{b}_1 = \hat{c}_1 \\ \dots \\ \hat{a}_7 \cdot \hat{b}_7 = \hat{c}_7 \\ \hat{a}_{15} \cdot \hat{b}_{15} = \hat{c}_{15} \end{array} \Longrightarrow \begin{array}{c} \hat{e}_0 + \mathbb{i}\hat{o}_0 \\ \hat{e}_4 + \mathbb{i}\hat{o}_4 \\ \hat{e}_2 + \mathbb{i}\hat{o}_2 \\ \hat{e}_6 + \mathbb{i}\hat{o}_6 \\ \hat{e}_1 + \mathbb{i}\hat{o}_1 \\ \hat{e}_5 + \mathbb{i}\hat{o}_5 \\ \hat{e}_3 + \mathbb{i}\hat{o}_3 \\ \hat{e}_7 + \mathbb{i}\hat{o}_7 \end{array} \underset{\text{ifft}_8}{\Longrightarrow} \begin{array}{c} c_0 + \mathbb{i}c_1 \\ c_2 + \mathbb{i}c_3 \\ c_4 + \mathbb{i}c_5 \\ c_6 + \mathbb{i}c_7 \\ c_8 + \mathbb{i}c_9 \\ c_{10} + \mathbb{i}c_{11} \\ c_{12} + \mathbb{i}c_{13} \\ c_{14} + \mathbb{i}c_{15} \end{array} \tag{5}$$

Denoting by $\bar{i}$ the result of flipping all bits in i but its lowest (set) bit, the $\hat{a}_i$ and $\hat{b}_i$ are recovered from the f_i, and the e_i and the o_i are recovered from the $\hat{c}_i$ by

$$\begin{aligned} f_i &= \hat{a}_i + \mathbb{i}\hat{b}_i, \quad \hat{c}_{i+0} = \hat{e}_i + \omega^i \hat{o}_i \\ \overline{f_{\bar{i}}} &= \hat{a}_i - \mathbb{i}\hat{b}_i, \quad \hat{c}_{i+8} = \hat{e}_i - \omega^i \hat{o}_i \end{aligned}.$$

The strategy illustrated in (5) is well suited for unbalanced multiplications. For balanced multiplications – and certainly for squaring – a right angle convolution is more effective as it outputs the lower and upper coefficients of the product in the real and imaginary components of the convolution with no need for zero padding. This right angle convolution on balanced multiplications, while having the same asymptotic complexity, is slightly faster than (5) as it avoids these complicated extra passes over the data, which add $O(n)$ to the run time.

3. INTEGER MULTIPLICATION

Figures 1, 2, 7 and 4 show timings for three integer multiplication algorithms:

- variable prime fft: between three and nine 50 bit primes are used along with a truncated recursive matrix algorithm [2]. Intended for large operands and multi-threading as this is a cache friendly algorithm with the possibility of processing each prime in parallel.
- four prime fft: four fixed 50 bit primes are used to combat the slow nature of chinese remaindering a variable number of primes at smaller sizes. Not intended for large operands as no effort has been made to be cache friendly.
- complex fft: 53 bit complex floating point with the real optimizations of Section 2.5. No answers here are provably correct, but coefficient sizes were taken up to about 75% of the observed failure points in Table 1. A Monte Carlo style algorithm can be devised with $O(n)$ overhead by adding heuristic checks on the output. No truncation is performed at all because the real optimizations complicate matters in the fft and the truncated ifft loses accuracy. The fft sizes instead include small odd multiples (i.e. $1, 3, 5$, see Section 2.3) of powers of two.

In each case we correctly implement gmp's low level integer multiplication function `mpn_mul` [2]. For each plotted value of n, the average, maximum, and minimum timings of integer multiplications where the product always has n bits and the smaller operand ranges between $n/2$ and $n/4$ bits are shown with three dots. The complex fft shows a rather large spread because in the balanced case ($n/2$ bits $\cdot$ $n/2$ bits) the right angle convolution is 10-20% faster than (5). However, the right angle convolution, as implemented, quickly becomes slower than (5) as the multiplication becomes imbalanced.

4. TRUNCATION

For the purposes of polynomial multiplication, zero padding the input (and output) sequences of the cyclic convolution to the next power-of-two size is clearly going to introduce ugly performance jumps right after these powers of two. However, it is possible to stay restricted to convolutions of length 2^k without these performance jumps by means of a *truncated fft*, which assumes certain portions of the input and output are zero. Here we use the approach of Harvey [2] modifed for the calculation sequence (3). If the desired output length is denoted by n we expect a runtime proportional to $n \log n$, so the more constant this ratio is the better the truncation is working. In order to graph this ratio, we adopt the notion from fftw [1] of a flop: an fft or ifft of length n is equivalent to $5n\log_2(n)$ floating point operations, regardless of the base ring. Figures 8, 9 and 10 show a graph of such a ratio for the three ffts in Section 3 with the input further truncated to length $n/2$.

- Truncating the complex fft had a disasterous effect on accuracy, therefore
- The complex fft is not truncated but rather working with lengths 2^k, $3 \cdot 2^{k-1}$, or $5 \cdot 2^{k-2}$.
- The timings for the four prime fft give the average per-prime timing.

[2] The timings for the one time setup of the twiddle tables is not included, and is only substantial for the base ring $\mathbb{C}$.

- The single prime fft should be slower than the amortized four prime time for smaller sizes because horizontal action within floating point registers is required when processing a single prime.
- The comparision with the general purpose library fftw is not quit fair because it has input format constraints and output ordering constraints that that are not required by the applications here. This comparison is included only to show the deleterious effects of an in order fft.

It seems that the speed of the fft over $\mathbb{C}$ is never more than a factor of two away from the speed over $\mathbb{F}_p$, with the latest processor showing the most impressive performance. What gives the complex fft its special speed in Section 3 is the fact that each data point has a real and imaginary part, so the fft is really processing twice as much data, and this is utilized by the real optimizations.

Figure 11 shows the corresponding inverted graph for FLINT's Schönhage–Strassen fft, where very large coefficients had to be used to ensure that the graph can continue to $n = 2^{18}$ while keeping the coefficient size constant.

5. CONCLUSIONS

In practice the complex fft does not produce incorrect answers if the input coefficients are reasonably restricted as in Table 1. Allowing for this Monte Carlo approach gives a win for the floating point units somewhere between Karastuba multiplication (2-way Toom-Cook) and 3-way Toom-Cook, hence the complicated higher order Toom-Cook algorithms simply do not need to be implemented. The following processor comparisons are clear from the graphs:

- Bulldozer (amd): this first attempt at an x86 fma architecture shows little benefit in a complex fft. Beats gmp by a factor of 2 at the largest size.
- Zen 2 (amd): this architecture improves floating point performance to the extent that a complex fft always wins in the relevant range. Beats gmp by a factor of 2.5 at the largest size.
- Coffee lake (intel): very impressive complex fft performance, though cache effects are noticeable when not mitigated by software. Beats gmp by a factor of 3 at the largest size.
- Zen 3 (amd): very impressive overall floating point performance, though similar relative performance of the complex fft to Zen 2. Beats gmp by a factor of 3.75 at the largest size.

The floating point units are operating at a small fraction (about 1/4) of their theoretical peak gflops numbers. Improving this figure to get closer to the theoretical peak seems a difficult task as ffts are mainly memory-bound and achieving decent pipeling of the basic operations in (1) or (3) is also difficult.

References

[1] M. Frigo and S.G. Johnson. The Design and Implementation of FFTW3. *Proceedings of the IEEE*, 93(2):216 –231, February 2005.

[2] David Harvey. A cache-friendly truncated fft, 2008.

Figure 1. n bit integer product on desktop Bulldozer

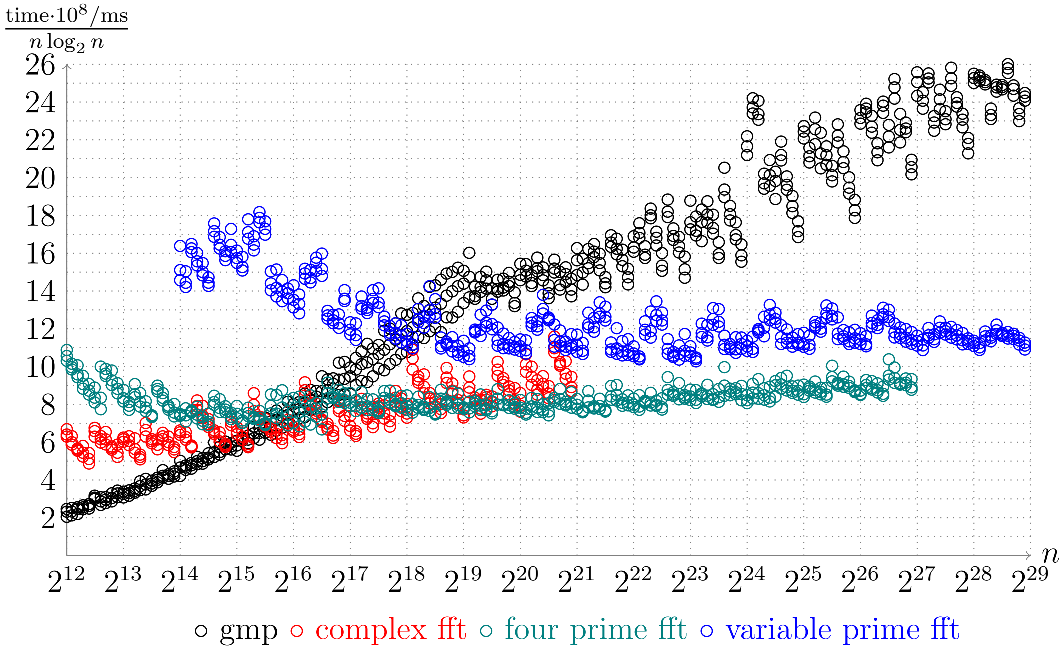


Figure 2. n bit integer product on mobile Zen 2

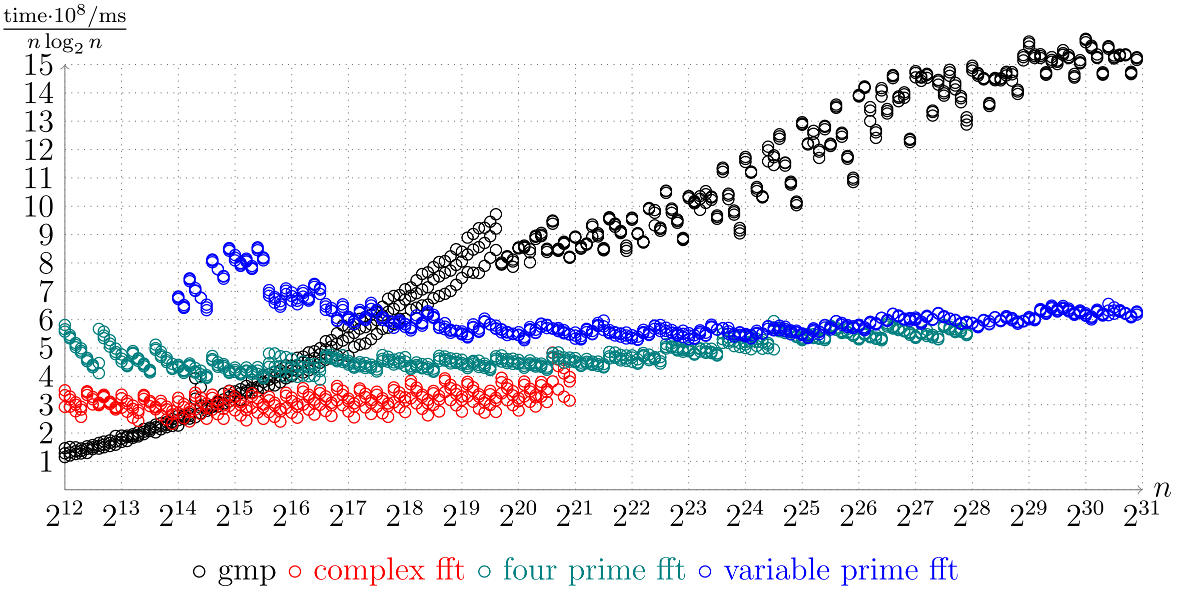


FIGURE 3. n bit integer product on desktop Coffee Lake

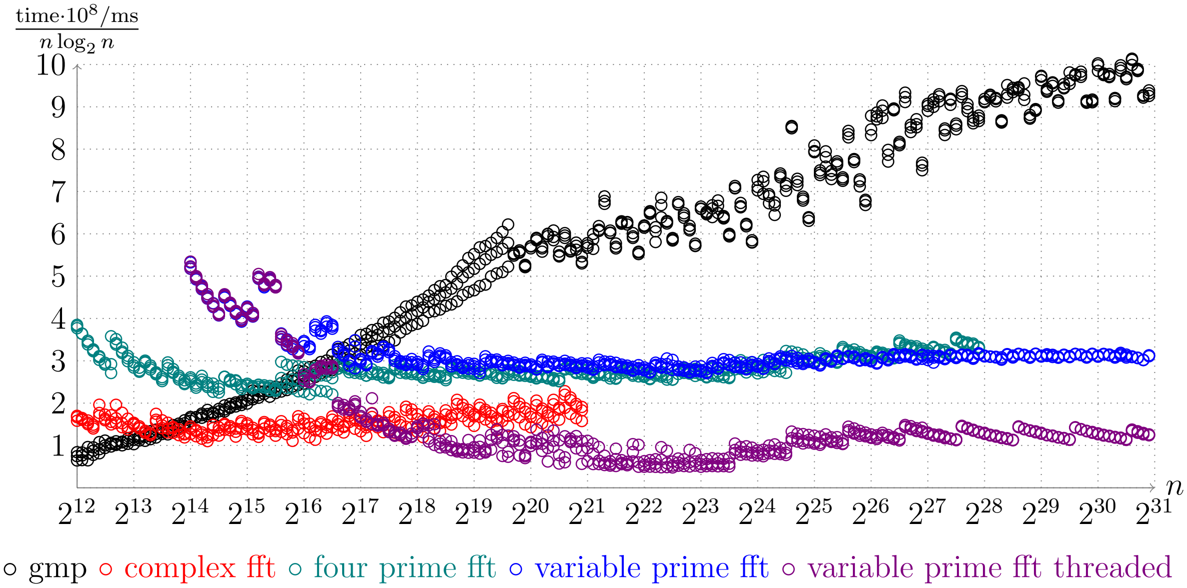


FIGURE 4. n bit integer product on Apple M1

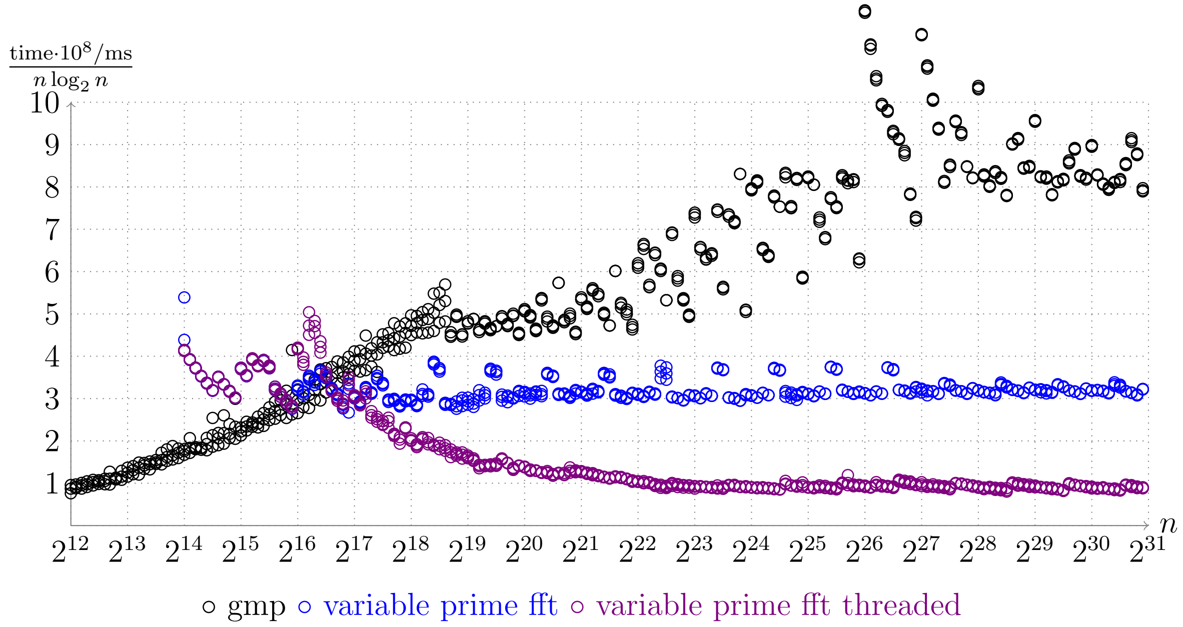


FIGURE 5. n bit integer square on desktop Coffee Lake

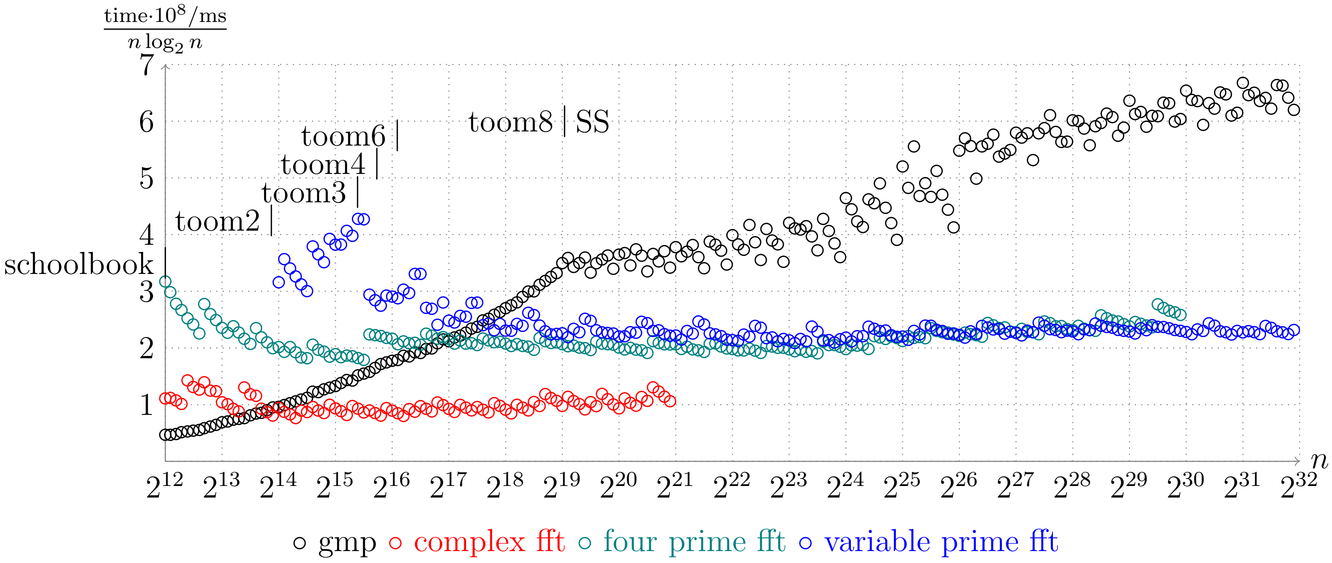


FIGURE 6. n bit integer product on mobile Zen 3 (AMD Ryzen 7 4800HS)

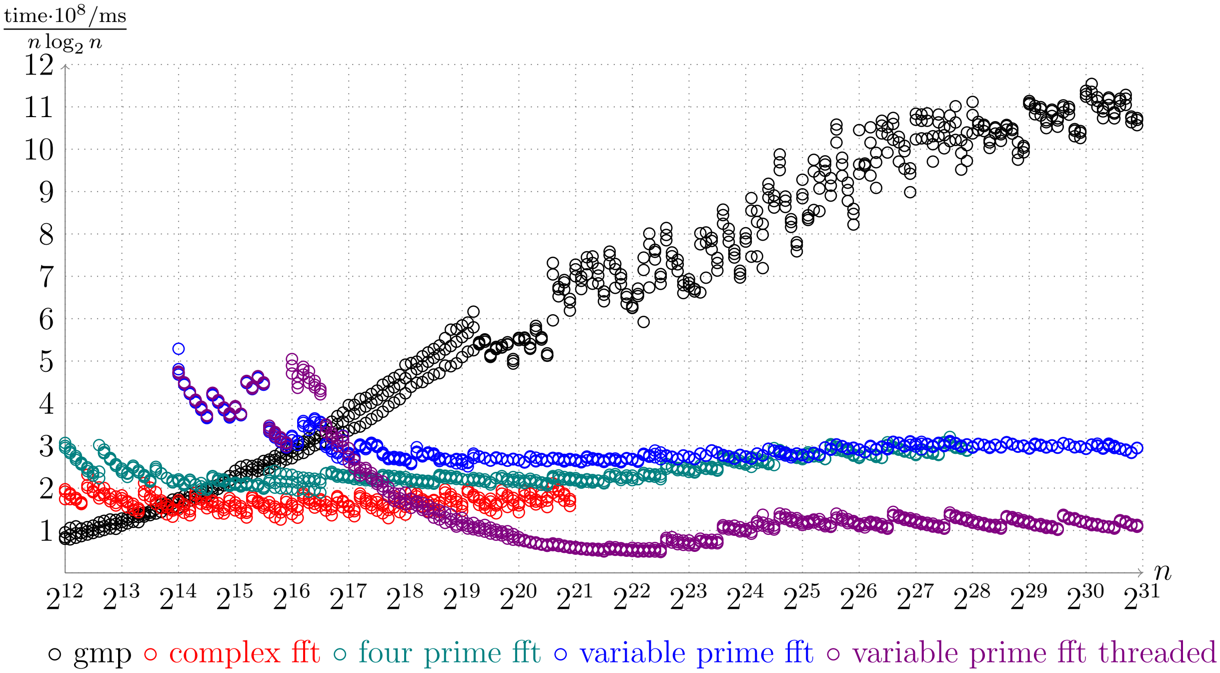


FIGURE 7. n bit integer product on mobile Zen 3 (AMD Ryzen 7 PRO 5850U)

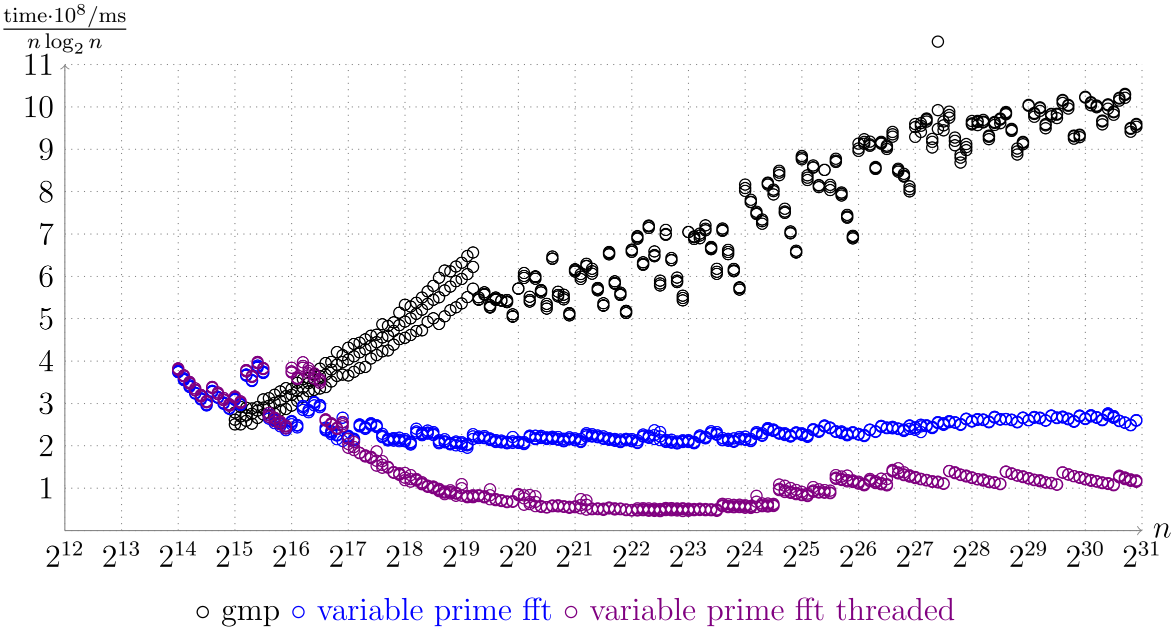


FIGURE 8. Truncation effectiveness in gflops on desktop Bulldozer

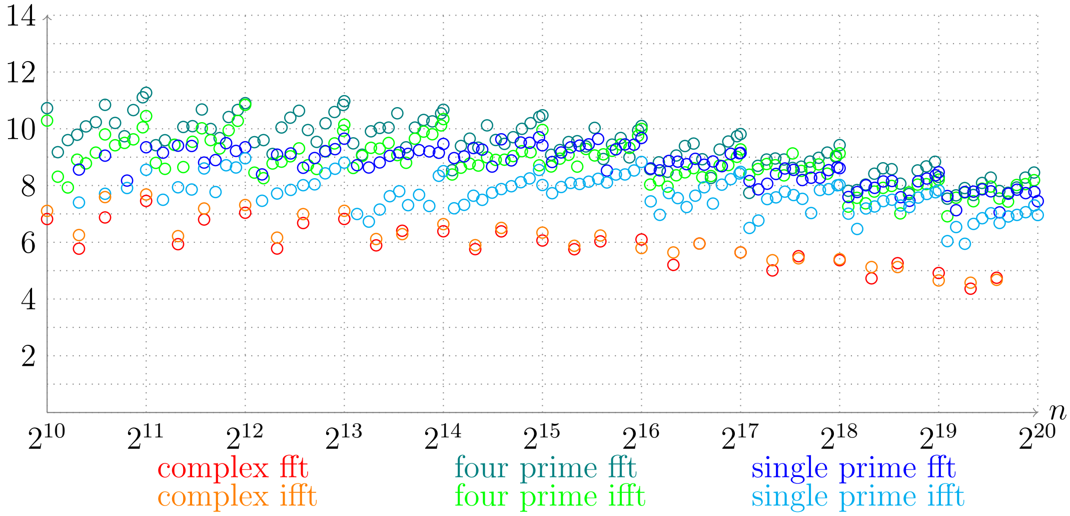


Figure 9. Truncation effectiveness in gflops on mobile Zen 2

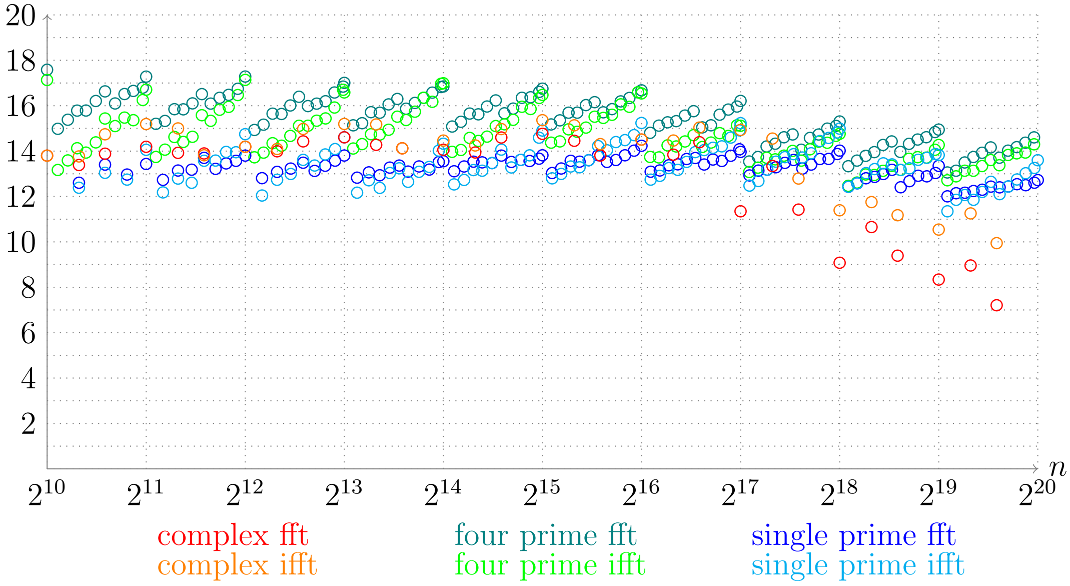


Figure 10. Truncation effectiveness in gflops on desktop Coffee Lake

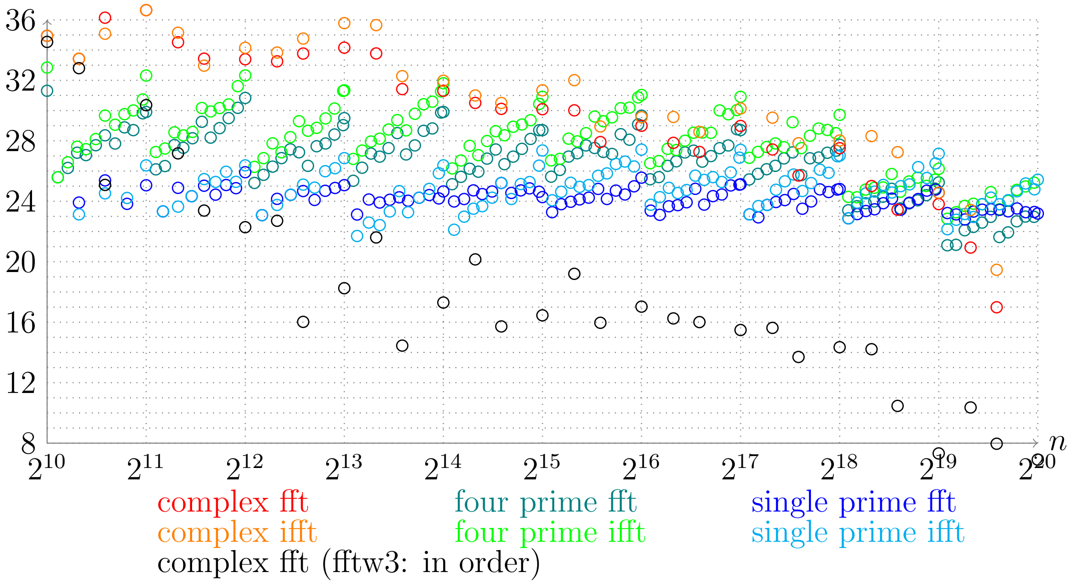
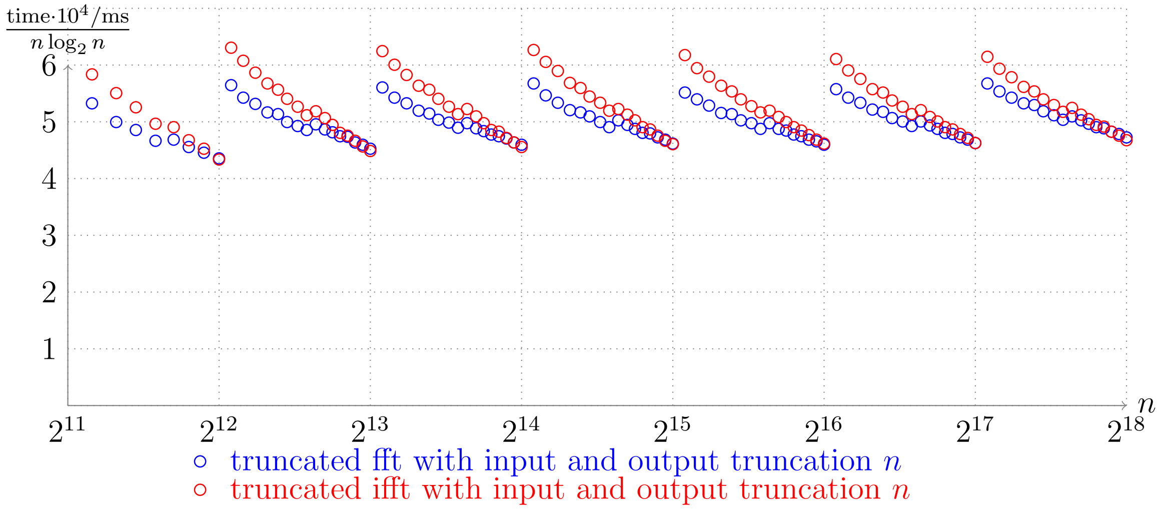


Figure 11. Truncation effectiveness of {i}`fft_mfa_truncate_sqrt2` with 2^{16} bit coefficients


$\frac{\text{time}\cdot 10^4/\text{ms}}{n \log_2 n}$
n
2^{11}
2^{12}
2^{13}
2^{14}
2^{15}
2^{16}
2^{17}
2^{18}
○ truncated fft with input and output truncation n
○ truncated ifft with input and output truncation n