CUBIC MODULAR EQUATIONS IN TWO VARIABLES

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Abstract. By adding certain equianharmonic elliptic sigma functions to the coefficients of the Borwein cubic theta functions, an interesting set of six two-variable theta functions may be derived. These theta functions invert the $F_1\left(\frac{1}{3};\frac{1}{3};\frac{1}{3};1|x,y\right)$ case of Appell's hypergeometric function and satisfy several identities akin to those satisfied by the Borwein cubic theta functions. The work of Koike et al. is extended and put into the context of modular equations, resulting in a simpler derivation of their results as well as several new modular equations for Picard modular functions. An application of these results is a new two-parameter family of solvable nonic equations.

1. Introduction

The F_1 function is defined for $|x|<1$ and $|y|<1$ by

$$F_1\left(a;b_1;b_2;c|x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}}\frac{x^m y^n}{m!\ n!}, \tag{1.1}$$

where $(a)_n=a(a+1)\cdots(a+n-1)$, and an analytic continuation for $\mathrm{Re}(a)>0$ and $\mathrm{Re}(c-a)>0$ is given by the integral representation

$$F_1\left(a;b_1;b_2;c|x,y\right)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b_1}(1-yt)^{-b_2}dt. \tag{1.2}$$

In [12] the striking identity

$$F\left(1-x^3,1-y^3\right)=\frac{3}{1+x+y}F\left(\left(\frac{1+\omega x+\bar{\omega}y}{1+x+y}\right)^3,\left(\frac{1+\bar{\omega}x+\omega y}{1+x+y}\right)^3\right) \tag{1.3}$$

was derived in connection with the common limit of a three term iteration. Here, $\omega=e^{2\pi i/3}$, $F(x,y)=F_1\left(\frac{1}{3};\frac{1}{3};\frac{1}{3};1|x,y\right)$, and the function F_1 as defined in (1.1) is the first of the four two-variable hypergeometric functions introduced by Appell [1]. We will show that such an identity is part of a larger class of identities and derive the next member of this class,

$$\begin{aligned}&F\left(\frac{x^3(y^2+3)(xy^2-3x-6y)}{(xy-3)^3(xy+3)},\frac{y^3(x^2+3)(yx^2-3y-6x)}{(xy-3)^3(xy+3)}\right)=\frac{xy-3}{xy-3x-3y-3}\\&\times F\left(\frac{(x^2+3)(y+3)^3(yx^2-3y-6x)}{(xy+3)(xy-3x-3y-3)^3},\frac{(y^2+3)(x+3)^3(xy^2-3x-6y)}{(xy+3)(xy-3x-3y-3)^3}\right).\end{aligned} \tag{1.4}$$

Due to the reduction formula

$$F_1(a;b_1,b_2;c|x,x)={}_2F_1\left(a,b_1+b_2;c|x\right),$$

the specialization of $x = y$ in (1.3) and (1.4) reduces them to transformations involving the one-variable function

$$F(x) = {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1|x\right).$$

Thus, (1.3) and (1.4) can be viewed as two-variable generalizations of modular equations arising from Ramanujan's theory of elliptic functions to base three (theory of signature three). For this reason, we first review several of the key results of this theory in Section 2 before stating the main results on two-variable generalizations in Section 3. Sections 4 and 5 introduce six Θ constants that are central to obtaining (1.3) and (1.4). Finally, Section 6 gives motivated and simple proofs of (1.3) and (1.4) based on identities of Θ functions, while Section 7 gives some applications of the modular equations contained in (1.3) and (1.4).

The author would like to thank Armin Straub for verifying identities (1.3) and (1.4) mechanically using the Holonomic Systems Approach in [13] and Tito Piezas III for pointing out Remark 7.7. Much thanks is also due to the anonymous referees in improving the presentation of this paper.

2. Definition of the one-variable cubic modular equations and statements of known results

In this section we will recall several important results on cubic modular equations. The hypergeometic function that is central to this theory is the Gauss hypergeometric function

$$F(x) := F(x, x) = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1|x\right),$$

as mentioned in the introduction. The defining relation of a cubic modular equation is

Definition 2.1. *The variables m, α, and β are said to be related by a one-variable cubic modular equation of degree $n \in \mathbb{N}$ when the simultaneous relations*

$$\begin{aligned} F(\beta)m &= F(\alpha) \\ F(1-\beta)m &= nF(1-\alpha) \end{aligned} \tag{2.1}$$

hold.

The variables α and β are known as the moduli and the variable m is known as the multiplier. Such modular equations are intimately related to the principle Hecke congruence subgroup $\Gamma_0(N)$, which is defined as

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv N \bmod N \right\}.$$

For example, from the theory of modular forms, we know that a one-variable cubic modular equation of degree n induces a algebraic relation between α and β of degree exactly

$$d(n) = [\Gamma(3) : \Gamma_0(3n)] = \frac{3n}{4} \prod_{\substack{p|3n \\ p \text{ prime}}} \left(1 + \frac{1}{p}\right), \tag{2.2}$$

and that m can be given as a rational function of α and β. Identities that make this algebraic relationship explicit are known as cubic modular equations, and two examples are given below in Theorems 2.3 and 2.4.

In [5] and [6], the Borweins introduced Θ functions defined as (set $q = e^{2\pi i\tau}$ and $\gamma = \frac{2+\omega}{3}$):

$$\begin{aligned} a(\tau) &= \sum_{\mu\in\mathbb{Z}[\omega]} q^{\mu\bar{\mu}}, \\ c(\tau) &= \sum_{\mu\in\mathbb{Z}[\omega]+\gamma} q^{\mu\bar{\mu}}, \\ b(\tau) &= \sum_{\mu\in\mathbb{Z}[\omega]} \omega^{\mu+\bar{\mu}} q^{\mu\bar{\mu}}. \end{aligned} \tag{2.3}$$

These functions play a central role in cubic modular equations because of the parameterizations given in Proposition 2.2, which are consequences of the Borweins' result,

$$a(\tau) = F\left(\frac{c(\tau)^3}{a(\tau)^3}\right) = F\left(1 - \frac{b(\tau)^3}{a(\tau)^3}\right).$$

A standard consequence of such a result in the theory of modular forms is that every modular form of weight k with respect to the group

$$\Gamma_1(3) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \begin{matrix} c \equiv 0 \bmod 3 \\ d \equiv 1 \bmod 3 \end{matrix} \right\}$$

is of the form $a(\tau)^k P(c(\tau)^3/a(\tau)^3)$, where P is a polynomial of degree not more than $k/3$. Thus $a(\tau)$ and $c(\tau)^3$ generate the space of modular forms with respect to $\Gamma_1(3)$.

Proposition 2.2 (Catalog of Borwein Θ function evaluations)**.** *Suppose that m, α, and β are related by a one-variable cubic modular equation of degree n. We then have the following table for converting identities among Borwein Θ functions to modular equations and vice-versa.*

$$\begin{aligned} a(\tau) &= z, & a(n\tau) &= z/m \\ c(\tau) &= \alpha^{1/3} z, & c(n\tau) &= \beta^{1/3} z/m, \\ b(\tau) &= (1-\alpha)^{1/3} z, & b(n\tau) &= (1-\beta)^{1/3} z/m, \end{aligned}$$

where

$$z = F(\alpha).$$

Directly from the definitions (2.3), we can derive the following identity between Borwein Θ functions and its conversion to a modular equation via Proposition 2.2.

$$a(3\tau) = a(\tau) + 2b(\tau) \quad \Leftrightarrow \quad \frac{3}{m} = 1 + 2(1-\alpha)^{1/3}. \tag{2.4}$$

The following result on cubic modular equations of degree 3 is equivalent to Theorem 7.4 in [4] and is directly equivalent to (2.4).

Theorem 2.3. *The following is a parameterization of the one-variable cubic modular equation of degree* 3.

$$\begin{aligned} \beta &= x^3, \\ \alpha &= 1 - \left(\frac{1-x}{1+2x}\right)^3, \\ m &= 1 + 2x. \end{aligned}$$

The following result on cubic modular equations of degree 4 is equivalent to Theorem 6.4 in [4] or [3, Ch. 33], where the parameter $p = \frac{2x}{3-x}$ was used.

Theorem 2.4. *The following is a parameterization of the one-variable cubic modular equation of degree* 4.

$$\beta = \frac{x^4\left(x^2-9\right)}{\left(x^2-3\right)^3},$$
$$\alpha = \frac{x(x+3)^3(x^2-9)}{\left(x^2-6x-3\right)^3},$$
$$m = \frac{x^2-6x-3}{x^2-3}.$$

Although Theorems 2.3 and 2.4 present the cubic modular equations of degrees 3 and 4, the degrees of the underlying algebraic relationship between α and β are $d(3) = 3$ and $d(4) = 6$, respectively, in accordance with (2.2).

3. Definition of the two-variable modular equations and statements of main results

For certain rational values of the parameters a, b_1, b_2, and c the Schwarz map associated with the F_1 function may be inverted by automorphic functions, and identities such as (1.3) and (1.4) provide modular equations for these automorphic functions. The series in (1.1) satisfies a system of partial differential equations given by [1, p. 182],

$$\begin{aligned}
x(1-x)f_{xx} &= ab_1 f + \frac{b_1(1-y)y}{x-y} f_y + \left((a+b_1+1)x - c - \frac{b_2(1-x)y}{x-y}\right) f_x,\\
y(1-y)f_{yy} &= ab_2 f + \frac{b_2(1-x)x}{y-x} f_x + \left((a+b_2+1)y - c - \frac{b_1 x(1-y)}{y-x}\right) f_y,\\
(x-y)f_{xy} &= b_2 f_x - b_1 f_y.
\end{aligned} \tag{3.1}$$

In fact, for general values of the parameters a, b_1, b_2 and c, the function $F_1(x,y)$ is the unique solution that is holomorphic at $(0,0)$ and takes the value 1 there. At any point in the complement of the singular locus $\Lambda = \{(x,y) \mid xy(1-x)(1-y)(x-y) = 0\}$ there is a basis of three holomorphic solutions. Let us fix a point w in $\mathbb{C}^2 \setminus \Lambda$ and set $\vec{\eta} = (\eta_0, \eta_1, \eta_2)$ to be a basis of solutions at w. For every path p in $\mathbb{C}^2 \setminus \Lambda$ with initial point w and terminal point, say, z, we may consider $\vec{\eta}_p(z)$, which is defined as the value of the solutions at the terminal point when analytically continued along the path p. When the terminal point z is the same as the initial point, the resulting value of $\vec{\eta}_p(w)$ must be related to the original value of $\vec{\eta}(w)$ by an element of $\mathrm{GL}_3(\mathbb{C})$ since the coefficients of (3.1) are single-valued on $\mathbb{C}^2 \setminus \Lambda$. This process gives a map from the fundamental group of $\mathbb{C}^2 \setminus \Lambda$ to $\mathrm{GL}_3(\mathbb{C})$, which is known as the monodromy representation of the system (3.1). A special case of the results in [9] and a nice survey in [14] is a condition describing the cases when the image of the monodromy representation is discrete and the Schwarz map is invertible. The Schwarz map Φ takes paths from w to z (up to equivalence under homotopy) to points in $\mathbb{C}P^2$ and is defined by

$$\Phi: \ (p,z) \mapsto [\vec{\eta}_p(z)].$$

The condition in [9, p. 66] and Theorem 3.1 of [14] is: provided all of the numbers $1-a$, $1+a-c$, b_1, b_2, $c-b_1-b_2$ are rational numbers in the interval $(0,1)$, all of the numbers

$$\begin{array}{ccccc} 1-b_1-b_2 & a-b_1 & b_1-c+1 & c-a-b_1 & b_1+b_2-a \\ c-1 & a-b_2 & b_2-c+1 & c-a-b_2 & b_1+b_2+a-c \end{array}$$

must either be reciprocals of integers or non-positive numbers. In what follows, we will set $a=b_1=b_2=\frac{1}{3}$, $c=1$, and $\omega=e^{2\pi i/3}$ and consider the basis of solutions

$$\begin{aligned} \eta_0(x,y) &= F\left(x,y\right), \\ \eta_1(x,y) &= F\left(1-x,1-y\right), \\ \eta_2(x,y) &= (-x)^{-1/3}F\left(\frac{y}{x},\frac{1}{x}\right)-(-y)^{-1/3}F\left(\frac{x}{y},\frac{1}{y}\right). \end{aligned} \tag{3.2}$$

These solutions can also be obtained as integral periods via the integral representation (1.2),

$$\begin{aligned} \eta_0(x,y) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b_1}(1-yt)^{-b_2}\,dt, \\ \eta_1(x,y) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_{-\infty}^0 (-t)^{a-1}(1-t)^{c-a-1}(1-xt)^{-b_1}(1-yt)^{-b_2}\,dt, \\ \eta_2(x,y) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_{\frac{1}{x}}^{\frac{1}{y}} t^{a-1}(t-1)^{c-a-1}(xt-1)^{-b_1}(1-yt)^{-b_2}\,dt, \end{aligned} \tag{3.3}$$

for $a=b_1=b_2=\frac{1}{3}$, and $c=1$. Whenever the series defining the $\eta_i(x,y)$ do not converge, the values of the $\eta_i(x,y)$ should be computed via the integral representations (3.3). We also note that if $0<y<x<1$, then the $\eta_i(x,y)$ are positive quantities.

The class of identities we seek to establish has the form given in Definition 3.1.

Definition 3.1. *The variables m, α_1, α_2, β_1, and β_2 are said to be related by a two-variable cubic modular equation of degree $\nu\in\mathbb{Z}[\omega]$ when the simultaneous relations*

$$\begin{gathered} F(\beta_1,\beta_2)m = F(\alpha_1,\alpha_2) \\ F(1-\beta_1,1-\beta_2)m = \nu\bar{\nu}F(1-\alpha_1,1-\alpha_2) \\ \left(\frac{F\left(\frac{\beta_2}{\beta_1},\frac{1}{\beta_1}\right)}{(-\beta_1)^{\frac{1}{3}}}-\frac{F\left(\frac{\beta_1}{\beta_2},\frac{1}{\beta_2}\right)}{(-\beta_2)^{\frac{1}{3}}}\right)m = \nu\left(\frac{F\left(\frac{\alpha_2}{\alpha_1},\frac{1}{\alpha_1}\right)}{(-\alpha_1)^{\frac{1}{3}}}-\frac{F\left(\frac{\alpha_1}{\alpha_2},\frac{1}{\alpha_2}\right)}{(-\alpha_2)^{\frac{1}{3}}}\right) \end{gathered} \tag{3.4}$$

hold.

Note that we have five variables and three relations in these modular equations, so there are a total of two degrees of freedom in choosing the variables m, α_1, α_2, β_1, and β_2. Note also that a one-variable cubic modular equation of degree $\nu\bar{\nu}$ may be obtained by specializing the two-variable modular equation.

Proposition 3.2. *A one-variable modular equation of degree $\nu\bar{\nu}$ among m, α and β may be obtained from a two-variable modular equation of degree ν among m, α_1, α_2, β_1 and β_2*

by means of the substitutions

$$\begin{aligned} \alpha_1 &\to \alpha, \quad \alpha_2 \to \alpha, \\ \beta_1 &\to \beta, \quad \beta_2 \to \beta, \\ m &\to m. \end{aligned}$$

As indicated by the examples (1.3) and (1.4), the relationship between m, α_1, α_2, β_1, and β_2 that is induced by a two-variable cubic modular equation appears to be algebraic with the degree of the underlying algebraic relationship increasing with $|\nu|$. Theorem 3.3 makes the precise observation that when the defining relations of a two-variable cubic modular equation hold, $\mathbb{C}(m, \alpha_1, \alpha_2, \beta_1, \beta_2)$ is an algebraic extension of $\mathbb{C}(\alpha_1, \alpha_2)$ with degree bounded by $24(\nu\bar{\nu})^{18}$. This is, of course, an extremely pessimistic upper bound on the degree, but is probably near the best one can do without considering the prime factors of ν in $\mathbb{Z}[\omega]$. The actual degree of this relationship is given by the index

$$d(\nu) := [\Gamma(\sqrt{-3}) : (D^{-1}\Gamma(\sqrt{-3})D) \cap \Gamma(\sqrt{-3})]$$

where D is the diagonal matrix $\mathrm{diag}(1, \nu\bar{\nu}, \nu)$ and the group $\Gamma(\sqrt{-3})$, which is the monodromy group of functions η_0, η_1 and η_2, is defined in (4).

Theorem 3.3. *Suppose that the variables m, α_1, α_2, β_1, and β_2 are related by a two-variable cubic modular equation of degree $\nu \in \mathbb{Z}[\omega]$. We then have:*

1. *Any element $X \in \mathbb{C}(m, \alpha_1, \alpha_2, \beta_1, \beta_2)$ is related to α_1 and α_2 by a polynomial equation $f(X) = 0$ where the coefficients of $f(X)$ are in $\mathbb{C}(\alpha_1, \alpha_2)$ and the degree of $f(X)$ is exactly $d(\nu)$. The polynomial $f(X)$ may have repeated roots, and we have the crude bound $d(\nu) \le 24(\nu\bar{\nu})^{18}$.*
2. *The polynomials $f(X)$ for the elements $X = \beta_1$ and $X = \beta_2$ do not have repeated roots in the cases $\nu = \sqrt{-3}$ and $\nu = 2$. The degrees in these cases are $d(\sqrt{-3}) = 9$, and $d(2) = 18$.*
3. *Given that β_1 and β_2 are algebraically related to α_1 and α_2, the multiplier m is then algebraically related to α_1 and α_2 by the explicit formula*

$$m^3 = (\nu^2\bar{\nu})\left(\frac{\partial\alpha_1}{\partial\beta_1}\frac{\partial\alpha_2}{\partial\beta_2} - \frac{\partial\alpha_1}{\partial\beta_2}\frac{\partial\alpha_2}{\partial\beta_1}\right)\frac{\beta_1^{2/3}(1-\beta_1)^{2/3}\beta_2^{2/3}(1-\beta_2)^{2/3}(\beta_1-\beta_2)^{2/3}}{\alpha_1^{2/3}(1-\alpha_1)^{2/3}\alpha_2^{2/3}(1-\alpha_2)^{2/3}(\alpha_1-\alpha_2)^{2/3}}.$$

The third part of Theorem 3.3 should be compared to the analogous result for one-variable modular equations. If m, α and β are related by a one-variable modular equation of degree n then the formula for the multiplier,

$$m^2 = n\frac{d\alpha}{d\beta}\frac{\beta(1-\beta)}{\alpha(1-\alpha)},$$

follows easily as a corollary from Entry 30 in Chapter 11 of [2],

$$\frac{d}{d\alpha}\left(\frac{2\pi}{\sqrt{3}}\frac{F(1-\alpha)}{F(\alpha)}\right) = -\frac{1}{\alpha(1-\alpha)F(\alpha)^2}.$$

The analogous result for the function F_1 (given, for example, in Lemma 2.4 of [16]) can be used to derive the formula for the multiplier in Theorem 3.3, but we will derive this result by using an identity of Θ functions in Section 5.

The first example of a two-variable modular equation is the three term iteration of [11] and [12]. If we start with three positive numbers $a_0 \geq b_0 \geq c_0$, and form three sequences according to ([12, p. 133])

$$\begin{aligned} a_{n+1} &= (a_n + b_n + c_n)/3, \\ b_{n+1}^3 + c_{n+1}^3 &= a_{n+1}(a_n b_n + a_n c_n + b_n c_n) - a_n b_n c_n, \\ b_{n+1}^3 - c_{n+1}^3 &= (a_n - b_n)(b_n - c_n)(c_n - a_n)/(3\sqrt{-3}), \end{aligned} \tag{3.5}$$

then there is a common limit that satisfies

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = a_0 \; F\left(1 - \frac{b_0^3}{a_0^3}, 1 - \frac{c_0^3}{a_0^3}\right)^{-1}.$$

The branches of the cube roots used to obtain b_{n+1} and c_{n+1} are not trivial, but they can be chosen so that even-indexed terms in the sequences are all real numbers. The fact that the common limit of the three term iteration can be expressed as the reciprocal of an F_1 is equivalent to a modular equation for F_1, which can be stated as a modular equation of degree $\nu = \sqrt{-3}$.

Theorem 3.4 (Proposition 2.5 of [12])**.** *The following is a parameterization of the two-variable cubic modular equation of degree* $1 + 2\omega = \sqrt{-3}$*.*

$$\begin{aligned} \beta_1 &= x^3, & \beta_2 &= y^3, \\ \alpha_1 &= 1 - \left(\frac{1 + \bar{\omega}x + \omega y}{1 + x + y}\right)^3, & \alpha_2 &= 1 - \left(\frac{1 + \omega x + \bar{\omega}y}{1 + x + y}\right)^3, \\ m &= 1 + x + y. \end{aligned}$$

In the process of deriving this modular equation, we find that iteration (3.5) has the slightly more symmetric form,

$$\begin{aligned} a_{n+1} &= (a_n + b_n + c_n)/3, \\ (a_{n+1}^3 - b_{n+1}^3)^{1/3} &= (a_n + \bar{\omega}b_n + \omega c_n)/3, \\ (a_{n+1}^3 - c_{n+1}^3)^{1/3} &= (a_n + \omega b_n + \bar{\omega}c_n)/3. \end{aligned}$$

We will also establish a modular equation of higher degree.

Theorem 3.5. *The following is a parameterization of the two-variable cubic modular equation of degree* 2*.*

$$\begin{aligned} \beta_1 &= \frac{x^3(y^2+3)(xy^2-3x-6y)}{(xy-3)^3(xy+3)}, & \beta_2 &= \frac{y^3(x^2+3)(yx^2-3y-6x)}{(xy-3)^3(xy+3)}, \\ \alpha_1 &= \frac{(x^2+3)(y+3)^3(yx^2-3y-6x)}{(xy+3)(xy-3x-3y-3)^3}, & \alpha_2 &= \frac{(y^2+3)(x+3)^3(xy^2-3x-6y)}{(xy+3)(xy-3x-3y-3)^3}, \\ m &= \frac{xy-3x-3y-3}{xy-3}. \end{aligned}$$

The parameterizing variables can be given as

$$x = 1 - m\left(\frac{1-\alpha_1}{1-\beta_2}\right)^{1/3}, \quad y = 1 - m\left(\frac{1-\alpha_2}{1-\beta_1}\right)^{1/3}.$$

By Proposition 3.2, Theorems 3.4 and 3.5 reduce to Theorems 2.3 and 2.4 when $x = y$.

4. The six Θ functions

The Schwarz map mentioned in Section 3 was studied by Picard [18], where a complicated expression for the inverse of this Schwarz map was given. We will use a subsequent simplification of Picard's formula given by Shiga [19] as the basis for studying cubic modular equations in two variables. Shiga's simplification is presented in Proposition 5.4 in Section 5. We will introduce a set of Θ functions that are responsible for providing the automorphic functions that invert (3.2). These Θ functions are holomorphic functions defined on the set $\mathbb{B} = \{(\tau_1, \tau_2) \in \mathbb{C}P^2 : \tau_2\bar{\tau}_2 < \tau_1 + \bar{\tau}_1\}$.

Definition 4.1. *For $a = 2$ or $a = 6$, set*

$$T_a(u) = \frac{2\pi e^{\frac{\pi i}{72}(23-7a)}}{3^{3/8}\Gamma\left(\frac{1}{3}\right)^{3/2}} \sum_{z\in\mathbb{Z}+\frac{1}{a}} e^{\frac{\pi}{\sqrt{3}}u^2+2\pi iu(1+\frac{1}{a}-u)+\pi iz^2\omega}, \quad u \in \mathbb{C}.$$

The six Θ functions $\Theta_i : \mathbb{B} \to \mathbb{C}$ are then defined by

$$\begin{aligned}
\Theta_0(\tau_1,\tau_2) &= \sum_{\mu\in\mathbb{Z}[\omega]} q^{\mu\bar{\mu}} T_6(\mu\tau_2), & \Theta_3(\tau_1,\tau_2) &= \sum_{\mu\in\mathbb{Z}[\omega]} \omega^{\mu+\bar{\mu}} q^{\mu\bar{\mu}} T_6(\mu\tau_2),\\
\Theta_1(\tau_1,\tau_2) &= \sum_{\mu\in\mathbb{Z}[\omega]+\gamma} q^{\mu\bar{\mu}} T_6(\mu\tau_2), & \Theta_4(\tau_1,\tau_2) &= \sum_{\mu\in\mathbb{Z}[\omega]} \bar{\omega}^{\mu+\bar{\mu}} q^{\mu\bar{\mu}} T_6(\mu\tau_2),\\
\Theta_2(\tau_1,\tau_2) &= \sum_{\mu\in\mathbb{Z}[\omega]-\gamma} q^{\mu\bar{\mu}} T_6(\mu\tau_2), & \Theta_5(\tau_1,\tau_2) &= i\omega \sum_{\mu\in\mathbb{Z}[\omega]+\gamma} \bar{\omega}^{\omega\mu+\bar{\omega}\bar{\mu}} q^{\mu\bar{\mu}} T_2(\mu\tau_2),
\end{aligned}$$

where $q^a = \exp\left(\frac{-2\pi a}{\sqrt{3}}\tau_1\right)$ and $\gamma = \frac{2+\omega}{3}$.

The precise way that these Θ functions generalize the Borwein Θ functions is clearest in Lemma 4.5 below, where they are expanded in series about $\tau_2 = 0$, a location where the six functions reduce exactly to the functions $a(\tau)$, $b(\tau)$ and $c(\tau)$. The $\Gamma(1/3)$ factor in the definition of the functions $T_a(u)$ is added for convieince so that $T_6(0) = 1$. This is equivalent to the classical evaluation of the Dedekind η-function at ω, i.e.

$$|\eta(\omega)| = \frac{3^{1/8}}{2\pi}\Gamma\left(\frac{1}{3}\right)^{3/2},$$

which may be obtain from the so-called Chowla-Selberg formula [8, p. 110]. The three functions Θ_0, Θ_3 and Θ_4 were studied in [12] and [19]. However, for the purpose of deriving modular equations, it seems most natural to introduce the complete set of six functions. The Θ functions may be related to Riemann's Θ function of zero argument as follows. For $(\tau_1, \tau_2) \in \mathbb{B}$, define Ω to be a 3×3 symmetric matrix with positive definite imaginary part as

$$\Omega(\tau_1,\tau_2) = \frac{1}{\bar{\omega}-\omega}\begin{pmatrix} 2\tau_1-\tau_2^2 & (\omega-\bar{\omega})\tau_2 & \tau_1+\bar{\omega}\tau_2^2 \\ (\omega-\bar{\omega})\tau_2 & 1-\bar{\omega} & (\omega-1)\tau_2 \\ \tau_1+\bar{\omega}\tau_2^2 & (\omega-1)\tau_2 & 2\tau_1-\omega\tau_2^2 \end{pmatrix}.$$

This matrix is calculated in [11, p. 208], where the specific matrix $-\Omega(-1/v, \omega u/v)^{-1}$ resulted from their choice of a basis. The Θ functions are then given by,

$$\Theta_0(\tau_1,\tau_2) = e^{\frac{125\pi i}{72}} c\,\Theta\begin{bmatrix} 0 & \frac{1}{6} & 0 \\ 0 & \frac{5}{6} & 0 \end{bmatrix}(\vec{0}),\quad \Theta_3(\tau_1,\tau_2) = e^{\frac{125\pi i}{72}} c\,\Theta\begin{bmatrix} 0 & \frac{1}{6} & 0 \\ \frac{2}{3} & \frac{5}{6} & \frac{1}{3} \end{bmatrix}(\vec{0}),$$
$$\Theta_1(\tau_1,\tau_2) = e^{\frac{125\pi i}{72}} c\,\Theta\begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \\ 0 & \frac{5}{6} & 0 \end{bmatrix}(\vec{0}),\quad \Theta_4(\tau_1,\tau_2) = e^{\frac{125\pi i}{72}} c\,\Theta\begin{bmatrix} 0 & \frac{1}{6} & 0 \\ \frac{1}{3} & \frac{5}{6} & \frac{2}{3} \end{bmatrix}(\vec{0}),$$
$$\Theta_2(\tau_1,\tau_2) = e^{\frac{125\pi i}{72}} c\,\Theta\begin{bmatrix} \frac{2}{3} & \frac{1}{6} & \frac{2}{3} \\ 0 & \frac{5}{6} & 0 \end{bmatrix}(\vec{0}),\quad \Theta_5(\tau_1,\tau_2) = e^{\frac{31\pi i}{24}} c\,\Theta\begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{2}{3} \end{bmatrix}(\vec{0}),$$

where

$$c = \frac{2\pi}{3^{3/8}\Gamma\left(\frac{1}{3}\right)^{3/2}},$$

and

$$\Theta\begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix}(\vec{z}) = \sum_{\vec{n}\in\mathbb{Z}^3} e^{\pi i(\vec{n}+\vec{a})\Omega(\vec{n}+\vec{a})^T + 2\pi i(\vec{z}+\vec{b})(\vec{n}+\vec{a})^T}. \tag{4.1}$$

The series in Definition 4.1 may be obtained by setting $\vec{n} = (x, z, y)$ in this sum. Since x and y range over all integers, the quantity $\mu = x - \omega^2 y$ ranges over all of $\mathbb{Z}[\omega]$. Finally, the sum on z may be evaluated by the functions $T_2(u)$ and $T_6(u)$.

The Picard modular group Γ is defined as

$$\Gamma = \{g \in \mathrm{GL}_3(\mathbb{Z}[\omega]) \mid \bar{g}^T H g = H\},\quad H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We will need two congruence subgroups of Γ. For any $\nu \in \mathbb{Z}[\omega]$, set $D = \mathrm{diag}(1, \nu\bar{\nu}, \nu)$ and

$$\Gamma(\nu) = \{g \in \Gamma \mid g \equiv I_{3\times 3} \mod \nu\},$$
$$\Gamma(\sqrt{-3}, \nu) = D^{-1}\Gamma(\sqrt{-3})D \cap \Gamma(\sqrt{-3}),$$

where $g \equiv I_{3\times 3} \mod \nu$ means that the entries of $g - I_{3\times 3}$ are all divisibe by ν. Any matrix $g = (g_{ij})_{ij} \in \Gamma$ acts on a point $(\tau_1, \tau_2) \in \mathbb{B}$ via

$$g : (\tau_1,\tau_2) \mapsto \left(\frac{g_{21} + g_{22}\tau_1 + g_{23}\tau_2}{g_{11} + g_{12}\tau_1 + g_{13}\tau_2}, \frac{g_{31} + g_{32}\tau_1 + g_{33}\tau_2}{g_{11} + g_{12}\tau_1 + g_{13}\tau_2}\right),$$

and the slash operator $|_{g,k}$ in weight k is defined as

$$f|_{g,k}(\tau_1,\tau_2) = \frac{1}{(g_{11} + g_{12}\tau_1 + g_{13}\tau_2)^k} f(g(\tau_1,\tau_2)).$$

The map $f \mapsto f|_{g,k}$ is referred to as the action of g on f, and the weight of this operator will always be clear from context.

The congruence subgroup $\Gamma(\sqrt{-3})$ is important because it is the monodromy group of (3.1) in the case $a = b_1 = b_2 = \frac{1}{3}$ and $c = 1$. According to [19, p. 331], we have $\Gamma/\Gamma(\sqrt{-3}) \simeq S_4$. Also according to [19, p. 328–332], a list of generators for $\Gamma(\sqrt{-3})/\{1, \omega, \omega^2\}$ can be given as $\{g_1, g_2, g_3, g_4, g_5\}$, and if this list is augmented by $\{g_6, g_7, g_8\}$, then it becomes a list of generators for Γ, where

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & \omega^2-\omega & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 0 & 0 \\ \omega^2-\omega & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$g_4 = \begin{pmatrix} 1 & 1-\omega & 1-\omega^2 \\ 0 & 1 & 0 \\ 0 & 1-\omega & 1 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 1 & 0 & 0 \\ 1-\omega & 1 & \omega^2-1 \\ \omega-1 & 0 & 1 \end{pmatrix},$$
$$g_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad g_7 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad g_8 = \begin{pmatrix} 1 & -\omega^2 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

In order to describe the automorphic behavior of the Θ functions under $\Gamma(\sqrt{-3})$, we need characters $\Gamma(\sqrt{-3}) \to \{1, \omega, \omega^2\}$ that describe the automorphic factors. For a given $\alpha \in \mathbb{Z}[\omega]$, let

$$\alpha \mod \sqrt{-3} := \begin{cases} 0, & \alpha \equiv 0 \mod \sqrt{-3} \\ 1, & \alpha \equiv 1 \mod \sqrt{-3} \\ 2, & \alpha \equiv 2 \mod \sqrt{-3} \end{cases},$$

and define eight characters $\chi_a : \Gamma(\sqrt{-3}) \to \{1, \omega, \omega^2\}$ by

$$\chi_0((g_{ij})_{ij}) = \exp \frac{2\pi i}{3} \left(\frac{2\,\mathrm{Im}(g_{11}g_{22} - g_{12}g_{21})}{\sqrt{3}} \right),$$
$$\chi_d((g_{ij})_{ij}) = \exp \frac{2\pi i}{3} \left(\frac{g_{11}+g_{22}+g_{33}}{-\sqrt{-3}} \mod \sqrt{-3} \right),$$
$$\chi_{kl}((g_{ij})_{ij}) = \exp \frac{2\pi i}{3} \left(\frac{g_{kl}}{-\sqrt{-3}} \mod \sqrt{-3} \right) \quad \text{for} \quad k \neq l.$$

There are two relations among these characters, $\chi_{23}\chi_{31} = 1$ and $\chi_{13}\chi_{32} = 1$, as can be verified on the generators g_1, g_2, g_3, g_4 and g_5 of $\Gamma(\sqrt{-3})$, and the character χ_d is the determinant map.

Lemma 4.2. *The action of the generators g_i in weight one on each of the Θ functions is summarized in the following table.*

f	$f\|_{g_1}$	$f\|_{g_2}$	$f\|_{g_3}$	$f\|_{g_4}$	$f\|_{g_5}$	$f\|_{g_6}$	$f\|_{g_7}$	$f\|_{g_8}$
Θ_0	Θ_0	Θ_0	Θ_0	Θ_0	Θ_0	Θ_0	Θ_0	Θ_2
Θ_1	Θ_1	Θ_1	$\omega\Theta_1$	Θ_1	$\omega\Theta_1$	Θ_2	Θ_3	Θ_0
Θ_2	Θ_2	Θ_2	$\omega\Theta_2$	Θ_2	Θ_2	Θ_1	Θ_4	Θ_1
Θ_3	Θ_3	$\omega\Theta_3$	Θ_3	$\omega\Theta_3$	Θ_3	Θ_4	Θ_1	$-\omega\Theta_4$
Θ_4	Θ_4	$\omega\Theta_4$	Θ_4	Θ_4	Θ_4	Θ_3	Θ_2	$-\omega\Theta_5$
Θ_5	$\omega\Theta_5$	$\omega\Theta_5$	$\omega\Theta_5$	$\bar{\omega}\Theta_5$	$\bar{\omega}\Theta_5$	$-\Theta_5$	$-\Theta_5$	$\omega\Theta_3$

For any $g \in \Gamma(\sqrt{-3})$, the action of the weight one operator $|_g$ on each of the Θ functions is given by

$$\begin{aligned} \Theta_0|_g &= \chi_0\Theta_0, & \Theta_3|_g &= \chi_0\chi_{12}\chi_{32}\Theta_3, \\ \Theta_1|_g &= \chi_0\chi_{21}\chi_{23}\Theta_1, & \Theta_4|_g &= \chi_0\chi_{12}\chi_{13}\Theta_4, \\ \Theta_2|_g &= \chi_0\chi_{21}\chi_{31}\Theta_2, & \Theta_5|_g &= \chi_0\chi_d\chi_{12}\chi_{21}\Theta_5. \end{aligned}$$

Proof. The action of the generators extendes the calculations in Lemma 4.2 of [19]. In each case we apply the general transformation formula for Θ constants ([15] or [17]). The remaining relations for any $g \in \Gamma(\sqrt{-3})$ may then be verified on the generators $g_1, \dots, g_5$ and $\omega I_{3\times 3}$. $\square$

We will now give the series expansions of the Θ functions about the point $(\tau_1, \tau_2) = (\infty, 0)$, where the local variables

$$q = \exp\frac{-2\pi}{\sqrt{3}}\tau_1, \quad z = \frac{\Gamma\left(\frac{1}{3}\right)^3}{2\sqrt{3}\pi}\tau_2 \tag{4.2}$$

will be used. When the Θ functions are expanded in powers of q in Lemma 4.4, the coefficients on powers of q are elliptic σ functions corresponding to the equianharmonic ($(g_2, g_3) = (0,1)$) case of the Weierstrass $\wp$ function. Finis [10] obtained several properties of these functions, including the result that $T_2(\nu u)$ is a homogenous polynomial in $T_6(u)$ and $T_6(-u)$ of degreee $\nu\bar{\nu}$, but for our purposes here, we just need the series exansions of these functions. Similary, we can expand the Θ functions in powers of z and write down each coefficient as a function of q. Lemma 4.5 says that the coefficients are essentially modular forms with respect to the group $\Gamma_1(3)$.

Lemma 4.3. *Let $\wp(z) = \frac{1}{z^2} + \frac{z^4}{28} + \cdots$ be the Weierstrass $\wp$ function that satisfies $\wp'(z)^2 = 4\wp(z)^3 - 1$, and let $\sigma(z) = z - \frac{z^7}{840} + \cdots$ be the Weierstrass σ function defined by $-\frac{d^2}{dz^2}\log\sigma(z) = \wp(z)$. Then, the series expansions of T_2 and T_6 are*

$$\begin{aligned} T_2\left(\frac{2\pi z}{\Gamma(1/3)^3}\right) &= \sigma(z) = z - \frac{z^7}{840} - \frac{z^{13}}{28828800} + O\left(z^{19}\right), \\ T_6\left(\frac{2\pi z}{\Gamma(1/3)^3}\right) &= -\frac{1}{2}\left(\sqrt{3} + \wp'\left(\frac{z}{\sqrt{-3}}\right)\right)\sigma\left(\frac{z}{\sqrt{-3}}\right)^3 \\ &= 1 - \frac{iz^3}{6} + \frac{z^6}{360} - \frac{iz^9}{45360} + O\left(z^{12}\right). \end{aligned}$$

Proof. From Equations (31) and (32) of [10], for any $\mu \in \mathbb{Z}[\omega]$, the quasi-periodicity relations,

$$\begin{aligned} T_2(u+\mu) &= e^{\frac{2\pi}{\sqrt{3}}\bar{\mu}(u-\omega\mu)}T_2(u), \\ T_6(u+\mu) &= e^{\frac{2\pi}{\sqrt{3}}\bar{\mu}(u-\omega\mu)}\bar{\omega}^{\mu+\bar{\mu}}T_6(u), \end{aligned} \tag{4.3}$$

hold, and the zero set of $T_2(u)$ is $\mathbb{Z}[\omega]$ while the zero set of $T_6(u)$ is $\mathbb{Z}[\omega] - 1/\sqrt{-3}$. Furthermore, the period lattice for the case $(g_2, g_3) = (0,1)$ of the Weierstrass $\wp$ elliptic functions is $\frac{\Gamma(1/3)^3}{2\pi}\mathbb{Z}[\omega]$. The first equality between T_2 and σ is the standard representation of the σ

function by Jacobian elliptic Θ functions. The transformations in (4.3) and the fact that the zeros set of T_2 is $\mathbb{Z}[\omega]$ imply that there are constants A and B with

$$\frac{T_6(\sqrt{-3}u)}{T_2(u)^3} - \frac{T_6(-\sqrt{-3}u)}{T_2(u)^3} = A + B\ \wp\left(\frac{\Gamma(1/3)^3}{2\pi}u\right),$$
$$\frac{T_6(\sqrt{-3}u)}{T_2(u)^3} + \frac{T_6(-\sqrt{-3}u)}{T_2(u)^3} = C\ \wp'\left(\frac{\Gamma(1/3)^3}{2\pi}u\right)$$

since these are, respectively, even and odd elliptic functions of orders at most three with respect to the period lattice $\mathbb{Z}[\omega]$. Lemma 9 of [10] implies that $A = -\sqrt{3}$ and $B = 0$, and the special value $T_6(0) = 1$ gives $C = -1$. $\square$

Lemma 4.4. *Set $\gamma = \frac{2+\omega}{3}$, and q and z as in (4.2). The six Θ functions have the series expansions*

$$\Theta_0(\tau_1,\tau_2) = 1 + \left(6 + \frac{9z^6}{20} + \cdots\right)q + \left(6 - \frac{243z^6}{20} + \cdots\right)q^3 + O\left(q^4\right),$$
$$\Theta_{\frac{1}{2}}(\tau_1,\tau_2) = \left(3 \pm \frac{z^3}{2} - \frac{z^6}{120} + \cdots\right)q^{\frac{1}{3}} + \left(3 \mp 4z^3 - \frac{8z^6}{15} + \cdots\right)q^{\frac{4}{3}} + O\left(q^{\frac{7}{3}}\right),$$
$$\Theta_{\frac{3}{4}}(\tau_1,\tau_2) = 1 + \left(-3 \mp \frac{9z^3}{2} - \frac{9z^6}{40} + \cdots\right)q + \left(6 - \frac{243z^6}{20} + \cdots\right)q^3 + O\left(q^4\right),$$
$$\Theta_5(\tau_1,\tau_2) = \left(3z + \frac{z^7}{280} + \cdots\right)q^{\frac{1}{3}} + \left(-6z - \frac{16z^7}{35} + \cdots\right)q^{\frac{4}{3}} + O\left(q^{\frac{7}{3}}\right).$$

Proof. These follow directly from Definition 4.1 and series expansions in Lemma 4.3. $\square$

Lemma 4.5. *Set q and z as in (4.2), and set $a = a(\tau)$, $b = b(\tau)$, $c = c(\tau)$ to be the Borwein cubic Θ functions where $q = e^{2\pi i\tau}$. We then have*

$$\Theta_0(\tau_1,\tau_2) = \sum_{k=0}^{\infty} aP_0^{2k}z^{6k} = a + \frac{ab^3c^3}{60}z^6 - \frac{19ab^3c^3\left(3a^6 - 4b^3c^3\right)}{1108800}z^{12} + \cdots,$$
$$\Theta_{\frac{1}{2}}(\tau_1,\tau_2) = \sum_{k=0}^{\infty} cP_{\frac{1}{2}}^{k}z^{3k} = c \pm \frac{b^3c}{6}z^3 - \frac{b^3c\left(a^3 + 2c^3\right)}{360}z^6 + \cdots,$$
$$\Theta_{\frac{3}{4}}(\tau_1,\tau_2) = \sum_{k=0}^{\infty} bP_{\frac{3}{4}}^{k}z^{3k} = b \mp \frac{bc^3}{6}z^3 + \frac{bc^3\left(2c^3 - 3a^3\right)}{360}z^6 + \cdots,$$
$$\Theta_5(\tau_1,\tau_2) = \sum_{k=0}^{\infty} bcP_5^{2k}z^{6k+1} = bcz + \frac{bc\left(a^6 - 6b^3c^3\right)}{840}z^7 + \cdots,$$

where, for each $0 \le i \le 5$,

$$P_i^k = \sum_{3m+3n=3k} C_{i,m}a^{3m}c^{3n}$$

for some rational coefficients $C_{i,m}$.

Proof. For

$$\begin{pmatrix} a & b \\ 3c & d \end{pmatrix} \in \Gamma_1(3),$$

consider the element

$$g = \begin{pmatrix} d & c\sqrt{-3} & 0 \\ -b\sqrt{-3} & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma(\sqrt{-3}).$$

By Lemma 4.2, we have

$$\begin{aligned} \Theta_0|_g &= \Theta_0, & \Theta_3|_g &= \omega^{-c}\Theta_3, \\ \Theta_1|_g &= \omega^{b}\Theta_1, & \Theta_4|_g &= \omega^{-c}\Theta_4, \\ \Theta_2|_g &= \omega^{b}\Theta_2, & \Theta_5|_g &= \omega^{b-c}\Theta_5. \end{aligned}$$

All of the assertions follow from these transformation formulas, since the action of g on the variables τ and z is

$$g \cdot (\tau, z) = \left(\frac{a\tau + b}{3c\tau + d}, \frac{z}{3c\tau + d} \right),$$

And this implied that the P_i^k are modular forms of weight $3k$ with respect to $\Gamma_1(3)$. Thus, they are polynomials in a^3 and c^3.

□

5. Picard Modular Forms and a Proof of Theorem 3.3

In order to prove that the modular equations in Definition 3.1 induce an algebraic relationship between (α_1, α_2) and (β_1, β_2), it is necessary to recall some background facts concerning Picard modular functions. For a subgroup G of Γ, we let $\overline{G}$ denote the subgroup of G consisting of matricies from G with determinant 1.

Lemma 5.1 (p. 349 of [19])**.** *For the group $\Gamma(\sqrt{-3})$, we have:*

(1) *The graded ring of holomorphic modular forms $f(\tau_1, \tau_2)$ of weight $3k$ that satisfy $f|_{g,3k} = f$ for all $g \in \overline{\Gamma(\sqrt{-3})}$ is $\mathbb{C}[\xi_0, \xi_1, \xi_2, \Delta]/(\Delta^3 - \xi_0\xi_1\xi_2(\xi_1 - \xi_0)(\xi_0 - \xi_2)(\xi_2 - \xi_1))$, where*

$$\begin{aligned} \xi_0(\tau_1, \tau_2) &= \Theta_0^3, \\ \xi_1(\tau_1, \tau_2) &= \Theta_1^3, \\ \xi_2(\tau_1, \tau_2) &= \Theta_2^3, \\ \Delta(\tau_1, \tau_2) &= \Theta_0\Theta_1\Theta_2\Theta_3\Theta_4\Theta_5. \end{aligned}$$

(2) *The graded ring of holomorphic modular forms $f(\tau_1, \tau_2)$ of weight $3k$ that satisfy $f|_{g,3k} = f$ for all $g \in \Gamma(\sqrt{-3})$ is $\mathbb{C}[\xi_0, \xi_1, \xi_2]$.*

(3) *The field of weight zero meromorphic modular functions with respect to $\Gamma(\sqrt{-3})$ is $\mathbb{C}(\frac{\Theta_1^3}{\Theta_0^3}, \frac{\Theta_2^3}{\Theta_0^3})$.*

The function Δ that appears in Lemma 5.1 is a basis for the one-dimensions space of holomorphic functions $f(\tau_1, \tau_2)$ satsifying $f|_{g,6} = \det(g) f$. Such a basis was given in [10] as

$$\left(\omega^2 \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_3} \right)^6 \Big|_{(z_1, z_2, z_2) = 0} \Theta((z_1, z_2, z_3)), \tag{5.1}$$

where $\Theta((z_1,z_2,z_3))$ is defined in (4.1) with $\vec{a}=\vec{b}=0$. It is thus amusing to notice the equality of these two representations of Δ. The definition given here as the product of all six Θ functions also seems more natural than (5.1).

Lemma 5.2. *We have the following identities.*

$$\begin{aligned}
\Theta_3(\tau_1,\tau_2)^3 &= \Theta_0(\tau_1,\tau_2)^3-\Theta_1(\tau_1,\tau_2)^3,\\
\Theta_4(\tau_1,\tau_2)^3 &= \Theta_0(\tau_1,\tau_2)^3-\Theta_2(\tau_1,\tau_2)^3,\\
\Theta_5(\tau_1,\tau_2)^3 &= \Theta_1(\tau_1,\tau_2)^3-\Theta_2(\tau_1,\tau_2)^3,\\
\Theta_0(\tau_1,\tau_2) &= F\left(\frac{\Theta_1(\tau_1,\tau_2)^3}{\Theta_0(\tau_1,\tau_2)^3},\frac{\Theta_2(\tau_1,\tau_2)^3}{\Theta_0(\tau_1,\tau_2)^3}\right)\quad \textit{for}\quad \Theta_0\ge\Theta_1\ge\Theta_2\ge 0,
\end{aligned}$$

Proof. The first three follow from Lemma 5.1. Each of the functions Θ_3^3, Θ_4^3, and Θ_5^3 is a modular form of weight 3 with respect to $\Gamma(\sqrt{-3})$. They must be linear combinations of Θ_0^3, Θ_1^3, and Θ_2^3 since these three functions span this space. The coefficients of these linear combinations may be found with the series expansions in Lemma 4.5. The last identity is equivalent to Corollary 2.1 in [11], and it must be considered a formal identity because $\eta_0(\lambda_1,\lambda_2)$ is in general a multi-valued function. On the branch fixed in (3.2), the equality

$$\Theta_0 = F\left(\frac{\Theta_1^3}{\Theta_0^3},\frac{\Theta_2^3}{\Theta_0^3}\right)$$

holds at least when $\Theta_0\ge\Theta_1\ge\Theta_2\ge 0$. □

Let $[q^nz^m]f(\tau_1,\tau_2)$ (or $[z^m]f(\tau_1,\tau_2)$) denote the coefficient of q^nz^m (or z^m) in the expansions of f in the variables (4.2). Thus, for example,

$$\begin{aligned}
[z^6q^1]\Theta_0 &= \frac{9}{20},\\
[z^6]\Theta_0 &= \frac{a(q)b^3(q)c^3(q)}{60}.
\end{aligned}$$

Lemma 5.3. *Set the expansion variables q and z as in* (4.2)*. Suppose that the holomorphic modular form $f(\tau_1,\tau_2)$ of weight $3k$ satisfies*

$$\begin{aligned}
f|_{g,3k} &= f,\\
[q^nz^m](f) &= 0.
\end{aligned}$$

for all $g\in\overline{\Gamma(\sqrt{-3})}$ and for all $m\le 3k$ and $n\le k+m/3$. In this case, $f(\tau_1,\tau_2)$ vanishes identically.

Proof. This follows from the fact given in Lemma 5.1 which states the space of holomorphic modular forms with respect to $\overline{\Gamma(\sqrt{-3})}$ is $\mathbb{C}(\xi_0,\xi_1,\xi_2,\Delta)$ modulo one relation for Δ^3, while the space of holomorphic modular forms with respect to $\Gamma_1(3)$ is $\mathbb{C}[a(\tau),c(\tau)^3]$.

As shown in the proof of Lemma 4.5, the coeffcient $[z^m](f)$ is a modular form of weight $m+3k$ with respect to $\Gamma_1(3)$. Since $a(\tau)=1+O(q)$ and $c(\tau)^3=q+O(q^2)$ generate this space, it has a basis of the form

$$\{q^n+O(q^{1+\lfloor k+m/3\rfloor})\}_{n=0}^{\lfloor k+m/3\rfloor}.$$

If all of the coefficients $[q^n z^m](f)$ vanish for $n \le k + m/3$, then it must be the case that the coefficient $[z^m](f)$ vanishes as a function of q.

Now write

$$f = f_0 + f_1\Delta + f_2\Delta^2,$$

where f_i is a modular form with respect to $\Gamma(\sqrt{-3})$ (not just $\overline{\Gamma(\sqrt{-3})}$) of weight $3k-6i$. The term $f_i\Delta^i$ collects those terms in the $(q,z)-$ expansion of f where the exponent on z is congruent to i modulo 3. Consider now what is means for $[z^0](f_0)$ to vanish (that is $f_0(\tau_1, 0) = 0$). According to the second part of Lemma 5.1, this means that f_0 is divisible by $\Theta_1^3 - \Theta_2^3$, that is, $(\Theta_1^3 - \Theta_2^3)^{-1} f_0$ is a modular form of weight $3k-3$. Continuing in this fashion, we deduce that $(\Theta_1^3 - \Theta_2^3)^{-k} f_0$ is a modular form of weight zero for $\Gamma(\sqrt{-3})$, that is, a constant. Since $[z^{3k}](f) = 0$, this constant must be zero. The forms $f_1(\tau_1,\tau_2)$ and $f_2(\tau_1,\tau_2)$ may be shown to vanish by similar arguments. $\square$

Proposition 5.4 (Catalog of Θ function evaluations)**.** *Suppose that m, α_1, α_2 β_1 and β_2 are related by a two-variable cubic modular equation of degree ν. We then have the following table for converting identities among Θ functions to modular equations and vice-versa.*

$$\begin{aligned}
\Theta_0(\tau_1,\tau_2) &= z, & \Theta_0(\nu\bar\nu\tau_1,\nu\tau_2) &= z/m\\
\Theta_1(\tau_1,\tau_2) &= \alpha_1^{1/3} z, & \Theta_1(\nu\bar\nu\tau_1,\nu\tau_2) &= \beta_1^{1/3} z/m,\\
\Theta_2(\tau_1,\tau_2) &= \alpha_2^{1/3} z, & \Theta_2(\nu\bar\nu\tau_1,\nu\tau_2) &= \beta_2^{1/3} z/m,\\
\Theta_3(\tau_1,\tau_2) &= (1-\alpha_1)^{1/3} z, & \Theta_3(\nu\bar\nu\tau_1,\nu\tau_2) &= (1-\beta_1)^{1/3}\ z/m\\
\Theta_4(\tau_1,\tau_2) &= (1-\alpha_2)^{1/3} z, & \Theta_4(\nu\bar\nu\tau_1,\nu\tau_2) &= (1-\beta_2)^{1/3} z/m,\\
\Theta_5(\tau_1,\tau_2) &= (\alpha_1-\alpha_2)^{1/3} z, & \Theta_5(\nu\bar\nu\tau_1,\nu\tau_2) &= (\beta_1-\beta_2)^{1/3} z/m,
\end{aligned}$$

where

$$z = F(\alpha_1,\alpha_2).$$

Proof. Set

$$\alpha_1 = \frac{\Theta_1(\tau_1,\tau_2)^3}{\Theta_0(\tau_1,\tau_2)^3}, \quad \alpha_2 = \frac{\Theta_2(\tau_1,\tau_2)^3}{\Theta_0(\tau_1,\tau_2)^3} \tag{5.2}$$

and assume that $0 < \alpha_2 < \alpha_1 < 1$. The last equality of Lemma 5.2 states that

$$\Theta_0(\tau_1,\tau_2) = F(\alpha_1,\alpha_2). \tag{5.3}$$

Replacing $(\tau_1,\tau_2) \to g_7(\tau_1,\tau_2)$ in (5.3) gives

$$\tau_1\Theta_0(\tau_1,\tau_2) = F(1-\alpha_1, 1-\alpha_2). \tag{5.4}$$

Next, replacing $(\tau_1,\tau_2) \to g_4 g_8 g_6 g_1^2(\tau_1,\tau_2)$ and $(\tau_1,\tau_2) \to g_8^{-1} g_6(\tau_1,\tau_2)$ in (5.3) gives, respectively,

$$\begin{aligned}
-\omega(1-\omega\tau_1+\omega\tau_2)\Theta_0(\tau_1,\tau_2) &= (-\alpha_1)^{-1/3} F\left(\frac{\alpha_2}{\alpha_1}, \frac{1}{\alpha_1}\right),\\
-\omega(1-\omega\tau_1-\tau_2)\Theta_0(\tau_1,\tau_2) &= (-\alpha_2)^{-1/3} F\left(\frac{\alpha_1}{\alpha_2}, \frac{1}{\alpha_2}\right).
\end{aligned} \tag{5.5}$$

Care must taken to ensure that the correct branch of the F function is used in (5.5). We have taken the standard one with the path of integration from 0 to 1 in (3.3) lying just below the branch cuts of the integrand. Equations (5.3), (5.4), and (5.5) may be combined to give the following method of inverting (5.2):

$$\begin{aligned}\tau_1 &= \frac{F(1-\alpha_1, 1-\alpha_2)}{F(\alpha_1,\alpha_2)},\\ \tau_2 &= \frac{(-\alpha_1)^{-\frac{1}{3}} F\left(\frac{\alpha_2}{\alpha_1}, \frac{1}{\alpha_1}\right) - (-\alpha_2)^{-\frac{1}{3}} F\left(\frac{\alpha_1}{\alpha_2}, \frac{1}{\alpha_2}\right)}{F(\alpha_1,\alpha_2)}.\end{aligned} \tag{5.6}$$

This agrees with fundamental inversion formula given in [18, p. 131] and [19, p. 327]. All of the conversions in this table follow from this inversion formula, the definition of the two-variable modular equation, and the identities given in Lemma 5.2. □

Modular equations of degrees $\bar{\nu}$ and $-\omega\nu$ are simply related to modular equations of degree ν, as the next Proposition demonstrates.

Proposition 5.5 (Reciprocation Process)**.** *If m, α_1, α_2 β_1 and β_2 are related by a two-variable cubic modular equation of degree ν, then a modular equation of degree $\bar{\nu}$ may be derived by the substitutions*

$$\begin{aligned}&\alpha_1 \to 1-\beta_1, \quad \alpha_2 \to 1-\beta_2,\\ &\beta_1 \to 1-\alpha_1, \quad \beta_2 \to 1-\alpha_2,\\ &m \to \frac{\nu\bar{\nu}}{m},\end{aligned}$$

and a modular equation of degree $-\omega\nu$ may be derived with the substitutions

$$\begin{aligned}&\alpha_1 \to \alpha_1, \quad \alpha_2 \to \alpha_2,\\ &\beta_1 \to \beta_2, \quad \beta_2 \to \beta_1,\\ &m \to m.\end{aligned}$$

Proof. Let τ_1 and τ_2 be defined as in Proposition 5.4. We may derive a modular equation of degree $\bar{\nu}$ by performing the substitution

$$(\tau_1, \tau_2) \to \left(\frac{1}{\nu\bar{\nu}\tau_1}, \frac{-\tau_2}{\nu\tau_1}\right).$$

The effects of this substitution on the Θ functions can be deduced by the entry for g_7 in Lemma 4.2. We may also derive a modular equation of degree $-\omega\nu$ by noticing that

$$\begin{aligned}\Theta_0(\nu\bar{\nu}\tau_1, -\omega\nu\tau_2) &= \Theta_0(\nu\bar{\nu}\tau_1, \nu\tau_2),\\ \Theta_1(\nu\bar{\nu}\tau_1, -\omega\nu\tau_2) &= \Theta_2(\nu\bar{\nu}\tau_1, \nu\tau_2),\\ \Theta_2(\nu\bar{\nu}\tau_1, -\omega\nu\tau_2) &= \Theta_1(\nu\bar{\nu}\tau_1, \nu\tau_2).\end{aligned}$$

Hence, the only effect is to interchange β_1 and β_2. □

Lemma 5.6. *For any $\nu \in \mathbb{Z}[\omega]$, $\Gamma(\nu)$ is a normal subgroup of Γ with $[\Gamma : \Gamma(\nu)] \leq (\nu\bar{\nu})^9$.*

Proof. The group $\Gamma(\alpha)$ is the kernel of the map $\phi : \Gamma \to \mathrm{GL}_3(\mathbb{Z}[\omega]/(\nu))$, which is obtained by reducing elements point-wise as

$$\phi : (g_{ij})_{ij} \mapsto (g_{ij} \mod \nu)_{ij}.$$

Thus, $|\Gamma/\Gamma(\nu)| \le |\,\mathrm{GL}_3(\mathbb{Z}[\omega]/(\nu))| \le (\nu\bar{\nu})^9$. $\square$

Proposition 5.7. *The first and second parts of Theorem 3.3 hold.*

Proof. By Proposition 5.4, the variables in a two-variable cubic modular equation are parameterized by $(\tau_1, \tau_2) \in \mathbb{B}$ as

$$\begin{aligned} \alpha_1 &= \frac{\Theta_1(\tau_1,\tau_2)^3}{\Theta_0(\tau_1,\tau_2)^3}, & \alpha_2 &= \frac{\Theta_2(\tau_1,\tau_2)^3}{\Theta_0(\tau_1,\tau_2)^3}, \\ \beta_1 &= \frac{\Theta_1(\nu\bar{\nu}\tau_1,\nu\tau_2)^3}{\Theta_0(\nu\bar{\nu}\tau_1,\nu\tau_2)^3}, & \beta_2 &= \frac{\Theta_2(\nu\bar{\nu}\tau_1,\nu\tau_2)^3}{\Theta_0(\nu\bar{\nu}\tau_1,\nu\tau_2)^3}, \\ m &= \frac{\Theta_0(\tau_1,\tau_2)}{\Theta_0(\nu\bar{\nu}\tau_1,\alpha\tau_2)}. \end{aligned}$$

Recall that D was the diagonal matrix $\mathrm{diag}(1, \nu\bar{\nu}, \nu)$. An element $X \in \mathbb{C}(m, \alpha_1, \alpha_2, \beta_2, \beta_2)$ is thus pushed forward to a modular function $X(\tau_1, \tau_2)$ with respect to the group

$$\Gamma(\sqrt{-3}, \nu) = (D^{-1}\Gamma(\sqrt{-3})D) \cap \Gamma(\sqrt{-3}).$$

Let us first obtain the bound on the index $d(\nu)$. We have,

$$\begin{aligned} d(\nu) = [\Gamma(\sqrt{-3}) : (D^{-1}\Gamma(\sqrt{-3})D) \cap \Gamma(\sqrt{-3})] &\le [\Gamma : \Gamma(\sqrt{-3})][\Gamma : (D^{-1}\Gamma D) \cap \Gamma] \\ &\le [\Gamma : \Gamma(\sqrt{-3})][\Gamma : \Gamma(\nu\bar{\nu})] \\ &\le 24(\nu\bar{\nu})^{18}. \end{aligned}$$

Now let $\Gamma(\sqrt{-3}) = \cup_i \Gamma(\sqrt{-3}, \nu)M_i$ be a decomposition of $\Gamma(\sqrt{-3})$ into right cosets with $M_0 = I$. The polynomial $f(x)$ is then

$$f(x) = \prod_i \left(x - X|_{DM_i}(\tau_1, \tau_2)\right).$$

By the third part of Lemma 5.1, the coefficients of $f(x)$ are rational functions of α_1 and α_2. From the factor with $i = 0$, we see that $f(x)$ has $x = X(\tau_1, \tau_2)$ as a root.

The second part of Theorem 3.3 follows from a straightforward calculation. The conjugates under the action of $\Gamma(\sqrt{-3})$ of

$$\beta_1 = \frac{\Theta_1(\nu\bar{\nu}\tau_1,\nu\tau_2)^3}{\Theta_0(\nu\bar{\nu}\tau_1,\nu\tau_2)^3}, \quad \beta_2 = \frac{\Theta_2(\nu\bar{\nu}\tau_1,\nu\tau_2)^3}{\Theta_0(\nu\bar{\nu}\tau_1,\nu\tau_2)^3},$$

which are $d(\nu)$ in number, are all distinct in the cases $\nu = \sqrt{-3}$ and $\nu = 2$. $\square$

In order to prove the third part of Theorem 3.3, it suffices to obtain a formula for the functional determinant

$$\frac{\partial(\tau_1, \tau_2)}{\partial(\alpha_1, \alpha_2)}$$

where (τ_1, τ_2) is related to (α_1, α_2) by the inversion formula (5.6). This is stated in terms of Θ functions in the following proposition. It is worth noting that such a determinant may be evaluated using a general formula for Wronskins given in Lemma 2.4 of [16], but we given a short self-contained proof based on the properties of the Θ functions.

Proposition 5.8.

$$\begin{vmatrix} \Theta_0^3 & \frac{\partial}{\partial\tau_1}\Theta_0^3 & \frac{\partial}{\partial\tau_2}\Theta_0^3 \\ \Theta_1^3 & \frac{\partial}{\partial\tau_1}\Theta_1^3 & \frac{\partial}{\partial\tau_2}\Theta_1^3 \\ \Theta_2^3 & \frac{\partial}{\partial\tau_1}\Theta_2^3 & \frac{\partial}{\partial\tau_2}\Theta_2^3 \end{vmatrix} = \Gamma\left(\frac{1}{3}\right)^3 \Theta_0^2\Theta_1^2\Theta_2^2\Theta_3^2\Theta_4^2\Theta_5^2$$

Proof. Let $f(\tau_1,\tau_2) = \Theta_0^9 \frac{\partial(\Theta_1^3/\Theta_0^3, \Theta_2^3/\Theta_0^3)}{\partial(\tau_1,\tau_2)}$ denote the functional determinant on the left hand side. Since we then have $\det(g) f|_{g,12} = f$ for all $g \in \Gamma(\sqrt{-3})$, the first part (but not the second part) of Lemma 5.1 applies. With this functional equation applied to $g = g_1$, we can observe that $f(\tau_1, \omega\tau_2) = \omega^2 f(\tau_1,\tau_2)$, hence f must contain only powers of z of the form z^{3n+2} and thus must be a constant multiple of $\Delta^2 = \Theta_0^2\Theta_1^2\Theta_2^2\Theta_3^2\Theta_4^2\Theta_5^2$ since it has weight 12. Using the series expansion in Lemmas 4.4 or 4.5, this constant is found to be $\Gamma\left(\frac{1}{3}\right)^3$. $\square$

Using the catalogue of evaluations in Proposition 5.4, we have

$$\begin{aligned} &\nu^2\bar\nu \frac{\partial(\alpha_1,\alpha_2)}{\partial(\beta_1,\beta_2)} = \frac{\partial(\alpha_1,\alpha_2)}{\partial(\tau_1,\tau_2)} \Big/ \frac{\partial(\beta_1,\beta_2)}{\partial(\nu\bar\nu\tau_1,\nu\tau_2)} \\ &= \frac{\Theta_0(\tau_1,\tau_2)^{-7}\Theta_1(\tau_1,\tau_2)^2\Theta_2(\tau_1,\tau_2)^2\Theta_3(\tau_1,\tau_2)^2\Theta_4(\tau_1,\tau_2)^2\Theta_5(\tau_1,\tau_2)^2}{\Theta_0(\nu\bar\nu\tau_1,\nu\tau_2)^{-7}\Theta_1(\nu\bar\nu\tau_1,\nu\tau_2)^2\Theta_2(\nu\bar\nu\tau_1,\nu\tau_2)^2\Theta_3(\nu\bar\nu\tau_1,\nu\tau_2)^2\Theta_4(\nu\bar\nu\tau_1,\nu\tau_2)^2\Theta_5(\nu\bar\nu\tau_1,\nu\tau_2)^2} \\ &= m^3 \frac{\alpha_1^{2/3}(1-\alpha_1)^{2/3}\alpha_2^{2/3}(1-\alpha_2)^{2/3}(\alpha_1-\alpha_2)^{2/3}}{\beta_1^{2/3}(1-\beta_1)^{2/3}\beta_2^{2/3}(1-\beta_2)^{2/3}(\beta_1-\beta_2)^{2/3}}, \end{aligned}$$

and so the third part of Theorem 3.3 is clear.

6. Proofs of the two-variable modular equations via theta functions

The goal of this section is to state and prove identities of Θ functions and then transfer them into modular equations via the catalog of evaluations in Proposition 5.4.

Theorem 6.1. *The following Θ function identities of degree $1+2\omega = \sqrt{-3}$ hold.*

$$\begin{aligned} 3\Theta_0(3\tau_1,(1+2\omega)\tau_2) &= \Theta_0(\tau_1,\tau_2) + \Theta_3(\tau_1,\tau_2) + \Theta_4(\tau_1,\tau_2), \\ 3\Theta_1(3\tau_1,(1+2\omega)\tau_2) &= \Theta_0(\tau_1,\tau_2) + \omega\Theta_3(\tau_1,\tau_2) + \bar\omega\Theta_4(\tau_1,\tau_2), \\ 3\Theta_2(3\tau_1,(1+2\omega)\tau_2) &= \Theta_0(\tau_1,\tau_2) + \bar\omega\Theta_3(\tau_1,\tau_2) + \omega\Theta_4(\tau_1,\tau_2). \end{aligned}$$

Proof. For $i = 0,1,2$, by the series expansions in Lemma 4.4, we have

$$\begin{aligned} &\Theta_0(\tau_1,\tau_2) + \omega^i\Theta_3(\tau_1,\tau_2) + \bar\omega^i\Theta_4(\tau_1,\tau_2) \\ &\qquad = \sum_{\mu\in\mathbb{Z}[\omega]} (1 + \omega^{i+\mu+\bar\mu} + \bar\omega^{i+\mu+\bar\mu}) q^{\mu\bar\mu} T_2(\mu\tau_2) \\ &\qquad = 3 \sum_{\substack{\mu\in\mathbb{Z}[\omega] \\ i+\mu+\bar\mu\equiv 0 \pmod 3}} q^{\mu\bar\mu} T_2(\mu\tau_2). \end{aligned}$$

The condition $i+\mu+\bar{\mu}\equiv 0 \pmod 3$ is satisfied exactly by the substitution $\mu \to \sqrt{-3}(\mu + i\frac{\omega-1}{3})$ where the new value of μ ranges over all of $\mathbb{Z}[\omega]$. Hence,

$$\begin{aligned}\Theta_0(\tau_1,\tau_2)+\omega^i\Theta_3(\tau_1,\tau_2)+\bar{\omega}^i\Theta_4(\tau_1,\tau_2)&\\ &= 3\sum_{\mu\in\mathbb{Z}[\omega]+i\frac{\omega-1}{3}} q^{3\mu\bar{\mu}}T_2(\sqrt{-3}\mu\tau_2)\\ &= 3\Theta_i(3\tau_1,(1+2\omega)\tau_2).\end{aligned}$$

□

Proof of Theorem 3.4. Applying Proposition 5.4 to the three equalities in Theorem 6.1 gives

$$\begin{aligned}3m^{-1} &= 1+\ (1-\alpha_1)^{1/3}+\ (1-\alpha_2)^{1/3},\\ 3m^{-1}\beta_1^{1/3} &= 1+\omega(1-\alpha_1)^{1/3}+\bar{\omega}(1-\alpha_2)^{1/3},\\ 3m^{-1}\beta_2^{1/3} &= 1+\bar{\omega}(1-\alpha_1)^{1/3}+\omega(1-\alpha_2)^{1/3}.\end{aligned}$$

Applying the first reciprocation process in Proposition 5.5 once and the second reciprocation process three times, we deduce that

$$\begin{aligned}m &= 1+\ \beta_2^{1/3}+\ \beta_1^{1/3},\\ m\ (1-\alpha_1)^{1/3} &= 1+\omega\beta_2^{1/3}+\bar{\omega}\beta_1^{1/3},\\ m\ (1-\alpha_2)^{1/3} &= 1+\bar{\omega}\beta_2^{1/3}+\omega\beta_1^{1/3},\end{aligned}$$

are also modular equations of degree $\sqrt{-3}$. The parameterizations in Theorem 3.4 are easily seen to be equivalent to these three relations. □

Theorem 6.2. *The following Θ function identities of degree 2 hold.*

$$\begin{aligned}\{\Theta_0(\tau_1,\tau_2)-\Theta_0(4\tau_1,2\tau_2)\}\,\Theta_5(\tau_1,\tau_2) &= +\{4\Theta_0(4\tau_1,2\tau_2)-\Theta_0(\tau_1,\tau_2)\}\,\Theta_5(4\tau_1,2\tau_2),\\ \{\Theta_2(\tau_1,\tau_2)-\Theta_1(4\tau_1,2\tau_2)\}\,\Theta_3(\tau_1,\tau_2) &= -\{4\Theta_1(4\tau_1,2\tau_2)-\Theta_2(\tau_1,\tau_2)\}\,\Theta_4(4\tau_1,2\tau_2),\\ \{\Theta_1(\tau_1,\tau_2)-\Theta_2(4\tau_1,2\tau_2)\}\,\Theta_4(\tau_1,\tau_2) &= -\{4\Theta_2(4\tau_1,2\tau_2)-\Theta_1(\tau_1,\tau_2)\}\,\Theta_3(4\tau_1,2\tau_2).\end{aligned}$$

Before commencing with the proof of these identities, we would like to give an indication of how they were discovered. We can observe with the help of Lemma 4.2 that the six functions

$$\begin{aligned}x_0 &= \frac{\Theta_0(\tau_1,\tau_2)}{\Theta_0(4\tau_1,2\tau_2)}, & x_1 &= \frac{\Theta_1(\tau_1,\tau_2)}{\Theta_2(4\tau_1,2\tau_2)}, & x_2 &= \frac{\Theta_2(\tau_1,\tau_2)}{\Theta_1(4\tau_1,2\tau_2)},\\ x_3 &= \frac{\Theta_3(\tau_1,\tau_2)}{\Theta_4(4\tau_1,2\tau_2)}, & x_4 &= \frac{\Theta_4(\tau_1,\tau_2)}{\Theta_3(4\tau_1,2\tau_2)}, & x_5 &= \frac{\Theta_5(\tau_1,\tau_2)}{\Theta_5(4\tau_1,2\tau_2)}\end{aligned}$$

are invariant under $\Gamma(\sqrt{-3},2)$. One can observe numerically that simple identities such as

$$(x_5+1)(x_0-1)=3$$

seem to hold, and this identity is equivalent to the first identity of Theorem 6.2.

Proof. We will prove only the first one, as the remaining two may be proven by identical methods. We wish to show that $f(\tau_1, \tau_2)$ vanishes identically where

$$f(\tau_1, \tau_2) = \{\Theta_0(\tau_1, \tau_2) - \Theta_0(4\tau_1, 2\tau_2)\}\,\Theta_5(\tau_1, \tau_2) - \{4\Theta_0(4\tau_1, 2\tau_2) - \Theta_0(\tau_1, \tau_2)\}\,\Theta_5(4\tau_1, 2\tau_2).$$

By Lemma 4.2, we can calculate that, for all $g \in \Gamma(\sqrt{-3}, 2)$,

$$f|_{g,2} = \chi_0(g)^2 \chi_{12}(g) \chi_{21}(g) \chi_d(g) f.$$

If $\Gamma(\sqrt{-3}) = \cup_{i=1}^{18} \Gamma(\sqrt{-3}, 2) M_i$ is a decomposition of $\Gamma(\sqrt{-3})$ into right cosets, let

$$F(\tau_1, \tau_2) = \prod_{i=1}^{18} f|_{M_i,2}.$$

We have $F|_{g,36} = F$ for all $g \in \Gamma(\sqrt{-3})$. If the condition

$$[q^n z^m](f) = 0 \tag{6.1}$$

of Lemma 5.3 holds for $f(\tau_1, \tau_2)$ for all $m \leq 3k$ and $n \leq k + m/3$, then it also holds for $F(\tau_1, \tau_2)$ since the other factors in the product are holomorphic. The requied coefficients in (6.1) (for $k = 12$) do indeed vanish, so $F(\tau_1, \tau_2)$ vanishes identically by Lemma 5.3. This means that $f(\tau_1, \tau_2)$ must also vanish identically. □

Proof of Theorem 3.5. Let us introduce parameters x and y defined by

$$m\left(\frac{1-\alpha_1}{1-\beta_2}\right)^{1/3} = 1 - x, \qquad m\left(\frac{1-\alpha_2}{1-\beta_1}\right)^{1/3} = 1 - y. \tag{6.2}$$

The last two equations in Theorem 6.2, when transfered using Proposition 5.4 and the definitions of x and y, become

$$m\left(\frac{\alpha_2}{\beta_1}\right)^{1/3} = 1 + \frac{3}{x}, \qquad m\left(\frac{\alpha_1}{\beta_2}\right)^{1/3} = 1 + \frac{3}{y}. \tag{6.3}$$

The first equation in Theorem 6.2 directly becomes

$$m\left(\frac{\alpha_1 - \alpha_2}{\beta_1 - \beta_2}\right)^{1/3} = \frac{3}{m-1} - 1. \tag{6.4}$$

We first solve (6.3) for β_1 and β_2 in terms of m, x, y, α_1 and α_2. We then substitute this solution for (β_1, β_2) into (6.2) and solve the resulting linear equations for (α_1, α_2) in terms of m, x, and y alone. The resulting solutions for (α_1, α_2) (and thus (β_1, β_2)) are

$$\begin{aligned}
\alpha_1 &= \frac{(y+3)^3\left(m^3 + (x-1)^3\right)}{m^3\left(x\left(x^2 - 3x + 3\right)y^3 + 9y^2 + 27y + 27\right)},\\
\alpha_2 &= \frac{(x+3)^3\left(m^3 + (y-1)^3\right)}{m^3\left(x^3 y\left(y^2 - 3y + 3\right) + 9x^2 + 27x + 27\right)},\\
\beta_1 &= \frac{x^3\left(m^3 + (y-1)^3\right)}{x^3 y\left(y^2 - 3y + 3\right) + 9x^2 + 27x + 27},\\
\beta_2 &= \frac{y^3\left(m^3 + (x-1)^3\right)}{x\left(x^2 - 3x + 3\right)y^3 + 9y^2 + 27y + 27}.
\end{aligned}$$

When these solutions are substituted into (6.4), we can solve for m in terms of x and y alone, and then obtain (α_1, α_2) and (β_1, β_2) in terms of x and y alone by substituting the obtained solution for m. When (6.4) is converted to an equation involving only m, x and y, there are six solutions for m given by

$$1-x, \quad \frac{x+3}{x}, \quad 1-y, \quad \frac{y+3}{y}, \quad \frac{xy+x+y-3}{x+y}, \quad \frac{xy-3x-3y-3}{xy-3}.$$

On account of the third part of Theorem 3.3, only the last one is correct. □

7. Applications: generators and a two parameter solvable nonic

In this section we will apply the modular equations of degree $\sqrt{-3}$ and 2 to deduce two consequences. First, we will deduce the structure of modular functions with respect to $\Gamma(\sqrt{-3}, \sqrt{-3})$ and $\Gamma(\sqrt{-3}, 2)$. This result should be compared with Shiga's result in the third part of Lemma 5.1, which gives the field of modular functions with respect to $\Gamma(\sqrt{-3})$. Second, we will show that the modular equation of degree 2 can be used to produce a two-parameter family of nonic equations whose Galois group is the Hessian group.

Theorem 7.1. *For the groups* $\Gamma(\sqrt{-3}, \sqrt{-3})$ *and* $\Gamma(\sqrt{-3}, 2)$*, we have*

(1) *The field of modular functions with respect to* $\Gamma(\sqrt{-3}, \sqrt{-3})$ *is*

$$\mathbb{C}\left(\frac{\Theta_3(\tau_1,\tau_2)}{\Theta_0(\tau_1,\tau_2)}, \frac{\Theta_4(\tau_1,\tau_2)}{\Theta_0(\tau_1,\tau_2)}\right) = \mathbb{C}\left(\frac{\Theta_1(3\tau_1,\sqrt{-3}\tau_2)}{\Theta_0(3\tau_1,\sqrt{-3}\tau_2)}, \frac{\Theta_2(3\tau_1,\sqrt{-3}\tau_2)}{\Theta_0(3\tau_1,\sqrt{-3}\tau_2)}\right).$$

(2) *The field of modular functions with respect to* $\Gamma(\sqrt{-3}, 2)$ *is*

$$\mathbb{C}\left(\frac{\Theta_3(\tau_1,\tau_2)}{\Theta_4(4\tau_1,2\tau_2)}, \frac{\Theta_4(\tau_1,\tau_2)}{\Theta_3(4\tau_1,2\tau_2)}\right) = \mathbb{C}\left(\frac{\Theta_2(\tau_1,\tau_2)}{\Theta_1(4\tau_1,2\tau_2)}, \frac{\Theta_1(\tau_1,\tau_2)}{\Theta_2(4\tau_1,2\tau_2)}\right)$$

Proof. First, each quotient of Θ functions is invariant under the corresponding $\Gamma(\sqrt{-3}, \nu)$ by Lemma 4.2. Set α_1, α_2, β_1 and β_2 as in Proposition 5.4. For each of the degrees $\nu = \sqrt{-3}$ and $\nu = 2$, the function β_1 (or β_2) has exactly $d(\nu)$ distinct conjugates under the action of $\Gamma(\sqrt{-3})$ (the polynomial $f(X)$ for the element $x = \beta_1$ in Theorem 3.3 does not have repeated roots). Since the modular equation for these degrees was rationally parameterized by some functions x and y, if we can demonstrate that x and y can be expressed as rational functions of the displayed quotients of Θ functions, it will follow that they generate the whole field. By the parameterizations of the Θ functions in Proposition 5.4, we have, for (1), where x and y are the parameters from Theorem 3.4,

$$\frac{\Theta_3(\tau_1,\tau_2)}{\Theta_0(\tau_1,\tau_2)} = \frac{1+\bar{\omega}x+\omega y}{1+x+y}, \quad \frac{\Theta_3(\tau_1,\tau_2)}{\Theta_0(\tau_1,\tau_2)} = \frac{1+\omega x+\bar{\omega}y}{1+x+y},$$

and

$$\frac{\Theta_1(3\tau_1,\sqrt{-3}\tau_2)}{\Theta_0(3\tau_1,\sqrt{-3}\tau_2)} = x, \quad \frac{\Theta_2(3\tau_1,\sqrt{-3}\tau_2)}{\Theta_0(3\tau_1,\sqrt{-3}\tau_2)} = y,$$

and for (2), where x and y are the parameters from Theorem 3.5,

$$\frac{\Theta_3(\tau_1,\tau_2)}{\Theta_4(4\tau_1,2\tau_2)} = 1-x, \quad \frac{\Theta_4(\tau_1,\tau_2)}{\Theta_3(4\tau_1,2\tau_2)} = 1-y,$$

and

$$\frac{\Theta_2(\tau_1,\tau_2)}{\Theta_1(4\tau_1,2\tau_2)} = 1+\frac{3}{x}, \quad \frac{\Theta_1(\tau_1,\tau_2)}{\Theta_2(4\tau_1,2\tau_2)} = 1+\frac{3}{y}.$$

Since in each case the pair of equations can be solved for x and y rationally in terms of the corresponding Θ function quotients, the proof is complete. □

The main result of this section is Theorem 7.2, which will be derived by a series of lemmas and propositions. This section culminates in Theorem 7.6, where Theorem 7.2 is proven and an explicit solution is given in radicals.

Theorem 7.2. *The Galois group of the splitting field of*

$$X^9 - 3X^8 + 4tX^6 - 6sX^5 - 6sX^4 + 4stX^3 - 3s^2X + s^2 = 0.$$

over the base field $\mathbb{C}(s,t)$ is the Hessian group of order 216. If the base field does not contain the cube roots of unity, for example $\mathbb{Q}(s,t)$, then the Galois group has order 432.

Lemma 7.3. *The group $\Gamma(\sqrt{-3},2)$ is an index 18 subgroup of $\Gamma(\sqrt{-3})$. A coset decomposition is given by $\Gamma(\sqrt{-3}) = \cup_i \Gamma(\sqrt{-3},2)M_i^{\pm}$ where $M_1 = I$,*

$$M_i^- = \begin{pmatrix} 1 & 2\sqrt{-3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} M_i^+,$$

and

$$M_2 = \begin{pmatrix} 1 & -2\omega-1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_3 = \begin{pmatrix} -3\omega-2 & -2\omega-1 & -3\omega-3 \\ 1-\omega & 1 & 1-\omega \\ \omega-1 & 0 & \omega \end{pmatrix},$$

$$M_4 = \begin{pmatrix} -3\omega-2 & -2\omega-1 & 3 \\ 1-\omega & 1 & 2\omega+1 \\ \omega-1 & 0 & -\omega-1 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 1 & -3\omega & \omega+2 \\ 0 & 1 & 0 \\ 0 & 1-\omega & 1 \end{pmatrix}$$

$$M_6 = \begin{pmatrix} -2 & -2\omega-1 & 0 \\ -2\omega-1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_7 = \begin{pmatrix} -3\omega-5 & -2\omega-1 & -3\omega-3 \\ -3\omega & 1 & 1-\omega \\ \omega-1 & 0 & \omega \end{pmatrix},$$

$$M_8 = \begin{pmatrix} -3\omega-2 & -2\omega-1 & 3\omega \\ 1-\omega & 1 & -\omega-2 \\ \omega-1 & 0 & 1 \end{pmatrix}, \quad M_9 = \begin{pmatrix} -3\omega-5 & -2\omega-1 & 3\omega \\ -3\omega & 1 & -\omega-2 \\ \omega-1 & 0 & 1 \end{pmatrix}.$$

Under the action of $\Gamma(\sqrt{-3})$ the cosets pairs are permuted according a group of order 216. The action of the generators is given in the following table.

	$M_1^{\pm}$	$M_2^{\pm}$	$M_3^{\pm}$	$M_4^{\pm}$	$M_5^{\pm}$	$M_6^{\pm}$	$M_7^{\pm}$	$M_8^{\pm}$	$M_9^{\pm}$
g_1	$M_1^{\pm}$	$M_2^{\pm}$	$M_4^{\pm}$	$M_8^{\pm}$	$M_9^{\mp}$	$M_6^{\pm}$	$M_5^{\mp}$	$M_3^{\pm}$	$M_7^{\pm}$
g_2	$M_2^{\pm}$	$M_1^{\mp}$	$M_5^{\pm}$	$M_9^{\mp}$	$M_3^{\mp}$	$M_6^{\pm}$	$M_8^{\pm}$	$M_7^{\mp}$	$M_4^{\pm}$
g_3	$M_1^{\pm}$	$M_6^{\pm}$	$M_7^{\pm}$	$M_5^{\mp}$	$M_4^{\pm}$	$M_2^{\mp}$	$M_3^{\mp}$	$M_9^{\pm}$	$M_8^{\mp}$
g_4	$M_3^{\pm}$	$M_5^{\pm}$	$M_2^{\mp}$	$M_7^{\mp}$	$M_1^{\pm}$	$M_6^{\pm}$	$M_9^{\pm}$	$M_4^{\pm}$	$M_8^{\pm}$
g_5	$M_1^{\pm}$	$M_8^{\pm}$	$M_5^{\mp}$	$M_3^{\pm}$	$M_7^{\mp}$	$M_9^{\pm}$	$M_4^{\mp}$	$M_6^{\mp}$	$M_2^{\pm}$

Proof. The 18 cosets follow from a straightforward calculation. This action of $\Gamma(\sqrt{-3})$ on the 9 pairs of right cosets defines a homomorphism $\rho : \Gamma(\sqrt{-3}) \to S_9$ (S_9 is the permutation group on 9 objects). The image in S_9 has the normal series

$$\mathrm{im}(\rho) = G_{216} \trianglerighteq G_{72} \trianglerighteq G_{18} \trianglerighteq G_9 \trianglerighteq G_1 = 1,$$

as we will proceed to show. In the remaining proof of this lemma let g_i denote the image of g_i under ρ (e.g. $g_1 = (348)(597)$). Since $g_2 = g_4^2$ and $g_3 = g_5^2$, the image of ρ is generated by g_1, g_4, and g_5. Define the groups G_i (of claimed order i) as

$$\begin{aligned} G_9 &= \langle g_4^2 g_5^2,\ g_4^2 g_5 g_4 \rangle = \langle (126)(384)(597), (137)(285)(496) \rangle \\ G_{18} &= \langle g_4^2,\ g_5^2,\ g_5 g_4 \rangle \\ G_{72} &= \langle g_4^2,\ g_5^2,\ g_5 g_4,\ g_5,\ g_1 g_5 g_1^{-1} \rangle \end{aligned}$$

That G_i does indeed have the claimed order i follows from the following computations involving the factor groups.

$$\begin{aligned} G_9 &\simeq \mathbb{Z}_3 \times \mathbb{Z}_3, \\ G_{18}/G_9 &= \{G_9,\ g_4^2 G_9\} \simeq \mathbb{Z}_2, \\ G_{72}/G_{18} &= \{G_{18},\ g_5 G_{18},\ g_1 g_5 g_1^{-1} G_{18},\ g_1^{-1} g_4 g_1 G_{18}\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2, \\ G_{216}/G_{72} &= \{G_{72},\ g_1 G_{72},\ g_1^2 G_{72}\} \simeq \mathbb{Z}_3. \end{aligned}$$

□

We now give the fundamental identities of Θ functions that are responsible for 'pairing up' the pairs of cosets in Lemma 4.2. This allows for an easy reduction of the degree of the modular equation from 18 to nine.

Proposition 7.4. *If $f^{\pm}(4\tau_1, 2\tau_2)$ denotes $f|_{DM_1^{\pm}}(\tau_1, \tau_2)$ where D is the diagonal matrix* $\mathrm{diag}(1, 4, 2)$*, then*

$$\begin{aligned} \Theta_0^+(4\tau_1, 2\tau_2) + {}& \Theta_0^-(4\tau_1, 2\tau_2) = \Theta_0(\tau_1, \tau_2)/2, \\ \Theta_1^+(4\tau_1, 2\tau_2) + {}& \Theta_1^-(4\tau_1, 2\tau_2) = \Theta_2(\tau_1, \tau_2)/2, \\ \Theta_2^+(4\tau_1, 2\tau_2) + {}& \Theta_2^-(4\tau_1, 2\tau_2) = \Theta_1(\tau_1, \tau_2)/2, \\ \Theta_3^+(4\tau_1, 2\tau_2) + {}& \omega\Theta_3^-(4\tau_1, 2\tau_2) = \Theta_4(\tau_1, \tau_2)/2, \\ \Theta_4^+(4\tau_1, 2\tau_2) + {}& \omega\Theta_4^-(4\tau_1, 2\tau_2) = \Theta_3(\tau_1, \tau_2)/2, \\ \Theta_5^+(4\tau_1, 2\tau_2) + {}& \omega\Theta_5^-(4\tau_1, 2\tau_2) = -\Theta_5(\tau_1, \tau_2)/2. \end{aligned} \tag{7.1}$$

Proof. We will only prove the first identity, as the remaining five may be proven similarly. By performing the substitution $(\tau_1, \tau_2) \to (1/(4\tau_1), -\tau_2/(2\tau_1))$ and using the fact that $\Theta_0|g_7 = \Theta_0$, this identity can be brought into the equivalent form

$$\Theta_0(\tau_1, \tau_2) + \Theta_0(\tau - \sqrt{-3}/2, \tau_2) = 2\Theta_0(4\tau_1, 2\tau_2).$$

From Lemma 4.4, where $q = e^{\frac{-2\pi}{\sqrt{3}}\tau_1}$,

$$\begin{aligned}
\Theta_0(\tau_1,\tau_2) + \Theta_0(\tau_1 - \sqrt{-3}/2, \tau_2) &= \sum_{\mu\in\mathbb{Z}[\omega]} q^{\mu\bar{\mu}} T_6(\mu\tau_2) + \sum_{\mu\in\mathbb{Z}[\omega]} (-1)^{\mu\bar{\mu}} q^{\mu\bar{\mu}} T_6(\mu\tau_2) \\
&= \sum_{\mu\in\mathbb{Z}[\omega]} (1 + (-1)^{\mu\bar{\mu}}) q^{\mu\bar{\mu}} T_6(\mu\tau_2) \\
&= 2 \sum_{\substack{\mu\in\mathbb{Z}[\omega],\\ \mu\bar{\mu}\in 2\mathbb{Z}}} q^{\mu\bar{\mu}} T_6(\mu\tau_2) \\
&= 2 \sum_{\mu\in\mathbb{Z}[\omega]} q^{4\mu\bar{\mu}} T_6(2\mu\tau_2) \\
&= 2\Theta_0(4\tau_1, 2\tau_2),
\end{aligned}$$

since $\mu\bar{\mu}$ is even if and only if $\mu \in 2\mathbb{Z}[\omega]$. □

The function

$$X(\tau_1,\tau_2) = \frac{4}{3}\left(\frac{\Theta_0^+(4\tau_1, 2\tau_2)}{\Theta_0(\tau_1,\tau_2)} - \frac{\Theta_0^-(4\tau_1, 2\tau_2)}{\Theta_0(\tau_1,\tau_2)}\right)^2$$

is clearly invariant under the action of M_1^-. It is also invariant under $\Gamma(\sqrt{-3})$ by Lemma 4.2. This means that it has only nine conjugates under the action of $\Gamma(\sqrt{-3})$, hence it should satisfy a polynomial of degree 9 with coefficients that are rational in the two functions

$$\alpha_1 = \frac{\Theta_1(\tau_1,\tau_2)^3}{\Theta_0(\tau_1,\tau_2)^3}, \quad \alpha_2 = \frac{\Theta_2(\tau_1,\tau_2)^3}{\Theta_0(\tau_1,\tau_2)^3},$$

by Lemma 5.1. In Table (7.2) and Table (7.3), we set $x_i = X(M_i^+(\tau_1,\tau_2))$ as the nine roots of our nonic and build a sequence of modular functions that can be used to solve for the x_i eventually in terms of radicals of α_1 and α_2. Since g_1, g_4, and g_5 generate the action of $\Gamma(\sqrt{-3})$ on the nine pairs of cosets given in Lemma 4.2, the action of these generators on the modular functions is also given.

(7.2)

elements	g_1 action	g_4 action	g_5 action
$x_i = X(M_i^+(\tau_1,\tau_2))$	(348)(597)	(1325)(4798)	(2869)(3574)
$3r_{i,j} = x_1 + \omega^j x_2 + \omega^{2j} x_6$ $+\omega^i(x_3 + \omega^j x_8 + \omega^{2j} x_4)$ $+\omega^{2i}(x_7 + \omega^j x_5 + \omega^{2j} x_9)$	$r_{i,j} \to r_{i+j,j}$	$r_{i,j} \to \omega^{j-i} r_{i-j,-i-j}$	$r_{i,j} \to r_{i-j,-i-j}$
$z_1 = r_{2,0} r_{2,1} r_{2,2}$ $z_2 = r_{0,1} r_{1,1} r_{2,1}$ $z_3 = r_{0,2} r_{1,1} r_{2,0}$ $z_4 = r_{0,1} r_{1,2} r_{2,0}$ $z_{\bar{i}} = g_4^2 z_i$	$(z_1 z_4 z_3)(z_{\bar{1}} z_{\bar{4}} z_{\bar{3}})$	$(z_1 z_{\bar{4}} z_{\bar{1}} z_4)(z_2 z_{\bar{3}} z_{\bar{3}} z_2)$	same as g_4

We also need another set of functions defined from the $r_{i,j}$ in Table (7.2). These functions will be necessary in finding in the quartic subfield of the spliting field of the nonic.

$$
\begin{array}{c|ccc}
\text{elements} & g_1 \text{ action} & g_4 \text{ action} & g_5 \text{ action} \\
\hline
\begin{array}{c} y_1 = r_{0,1}r_{0,2} - 1 \\ y_2 = r_{1,0}r_{2,0} - 1 \\ y_3 = r_{1,2}r_{2,1} - 1 \\ y_4 = r_{1,1}r_{2,2} - 1 \end{array} & (y_1y_4y_3) & (y_1y_4)(y_2y_3) & \text{same as } g_4 \\
\hline
\begin{array}{c} 4w_1 = +y_1 - y_2 - y_3 + y_4 \\ 4w_2 = -y_1 - y_2 + y_3 + y_4 \\ 4w_3 = -y_1 + y_2 - y_3 + y_4 \end{array} & \begin{array}{c} w_1 \to +w_2 \\ w_2 \to -w_3 \\ w_3 \to -w_1 \end{array} & \begin{array}{c} w_1 \to +w_1 \\ w_2 \to -w_2 \\ w_3 \to -w_3 \end{array} & \text{same as } g_4 \\
\hline
\begin{array}{c} u_1 = w_2w_3/w_1 \\ u_2 = w_3w_1/w_2 \\ u_3 = w_1w_2/w_3 \end{array} & (u_1u_2u_3) & \text{identity} & \text{identity}
\end{array}
\tag{7.3}
$$

Owing to the fact that Θ_0 is fixed by g_6 and g_7, it turns out that the x_i are permuted by g_6 and g_7 as well as $\Gamma(\sqrt{-3})$. This means that the coefficient field of the nonic satisfied by the x_i can be restricted to those functions that are invariant under not only $\Gamma(\sqrt{-3})$, but also g_6 and g_7. The reason for this simplification is partly due to the fact that the group $G = \mathrm{diag}(1,4,2)^{-1}\Gamma\,\mathrm{diag}(1,4,2)$ has index 18 in Γ (and $\Gamma(\sqrt{-3},2)$ is the intersection of G and $\Gamma(\sqrt{-3})$). This may be verified either by a direct enumeration, or by showing that $\Gamma = \Gamma(\sqrt{-3})G$, that is, each of the 24 equivalence classes in $\Gamma(1)/\Gamma(\sqrt{-3})$ are covered by some element of G. Thus, we have the same coset decomposition $\Gamma = \cup_i GM_i^{\pm}$. The action of g_6, g_7, and g_8 on these cosets is summarized in the following table.

	$M_1^{\pm}$	$M_2^{\pm}$	$M_3^{\pm}$	$M_4^{\pm}$	$M_5^{\pm}$	$M_6^{\pm}$	$M_7^{\pm}$	$M_8^{\pm}$	$M_9^{\pm}$
g_6	$M_1^{\pm}$	$M_2^{\pm}$	$M_3^{\mp}$	$M_4^{\mp}$	$M_5^{\mp}$	$M_6^{\pm}$	$M_7^{\mp}$	$M_8^{\mp}$	$M_9^{\mp}$
g_7	$M_6^{\mp}$	$M_2^{\pm}$	$M_9^{\mp}$	$M_7^{\mp}$	$M_8^{\pm}$	$M_1^{\mp}$	$M_4^{\mp}$	$M_5^{\pm}$	$M_3^{\mp}$
g_8	$M_8^{\pm}$	$M_7^{\mp}$	$M_9^{\mp}$	$M_3^{\mp}$	$M_4^{\mp}$	$M_6^{\pm}$	$M_1^{\pm}$	$M_2^{\pm}$	$M_5^{\pm}$

We must then verify that the functions $x_i(\tau_1,\tau_2) = X(M_i^+(\tau_1,\tau_2))$ are permuted accordingly by g_6 and g_7, which entails tediously checking the cube roots of unity in the automorphic factors to ensure that each is unity.

Lemma 7.5. *Set* $\alpha_1 = \Theta_1(\tau_1,\tau_2)^3/\Theta_0(\tau_1,\tau_2)^3$, *and* $\alpha_2 = \Theta_2(\tau_1,\tau_2)^3/\Theta_0(\tau_1,\tau_2)^3$. *The subfield of* $\mathbb{C}(\alpha_1,\alpha_2)$ *of functions that are also invariant under* g_6 *and* g_7 *is* $\mathbb{C}(s,t)$ *where*

$$s = (\alpha_1 - \alpha_2)^2, \quad t = 2(\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2).$$

Proof. Let F denote the required subfield of $\mathbb{C}(\alpha_1,\alpha_2)$. By Lemma 4.2, the action of g_6 and g_7 on α_1 and α_2 is given by

$$g_6(\alpha_1) = \alpha_2, \quad g_6(\alpha_2) = \alpha_1, \quad g_7(\alpha_1) = 1 - \alpha_1, \quad g_7(\alpha_2) = 1 - \alpha_2,$$

hence $\mathrm{Gal}(\mathbb{C}(\alpha_1,\alpha_2)/F) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. Since $[\mathbb{C}(\alpha_1,\alpha_2) : \mathbb{C}(s,t)] = 4$ by the defining equations for s and t, the equality $F = \mathbb{C}(s,t)$ holds. $\square$

By the first identity of Proposition 7.4, the function $X(\tau_1,\tau_2)$ can be rewritten as

$$X(\tau_1,\tau_2) = \frac{4}{3}\left(2\frac{\Theta_0(4\tau_1,2\tau_2)}{\Theta_0(\tau_1,\tau_2)} - \frac{1}{2}\right)^2 = \frac{1}{3}\left(\frac{4}{m}-1\right)^2,$$

where m is the multiplier of the associated modular equation of degree $\nu = 2$. The polynomial equation satisfied by X is found to be

$$X^9 - 3X^8 + 4tX^6 - 6sX^5 - 6sX^4 + 4stX^3 - 3s^2X + s^2 = 0. \tag{7.4}$$

In order to compute this equation, we note that the coefficient of X^{9-i} must be a polynomial in s and t of total degree no more than $i/3$. One can form such an equation as (7.4) with the coefficient of X^{9-i} as polynomials in s and t of total degree $\lfloor i/3 \rfloor$ with undetermined coefficients. Since s, t and X are given rationally in terms of x and y in Theorem 3.5, the undetermined coefficients may be found by specializing (x,y) to rational pairs and solving the resulting linear system.

In the calculations for explicitly solving (7.4), we will need the series expansions of the roots. This allows certain combinations of the roots to be identified as polynomials in s and t. There are three roots that can be expanded in an ordinary power series at $s=0$, namely,

$$R_3(r) := r + \frac{6r^2+6r-4t}{3r^2(3r^3-8r^2+8t)}s + O\left(s^2\right),$$

where r is any of the three roots of $r^3 - 3r^2 + 4t = 0$. The other six roots can be expanded in an Pisuex series at $s=0$ as

$$R_6(p) := ps^{1/3} + \frac{1}{4pt}s^{2/3} + \frac{(2p^3t+1)}{8t^2(2p^3+1)}s + O\left(s^{4/3}\right),$$

where p is any of the six roots of $4tp^6 + 4tp^3 + 1 = 0$. Consider the behavior of the roots in the neighborhood of $(s,t) = (0,1)$ with $s>0$ and $t<1$. There are three real roots and three pairs of complex conjugate roots. From the ordering of the cosets in the definition of the functions $x_i(\tau_1,\tau_2)$, we have the ordering of the roots

$$\begin{aligned}
&x_1 = R_3\left(2 - \frac{2\sqrt{1-t}}{\sqrt{3}} + \cdots\right), && x_2 = R_3\left(-1+\cdots\right), && x_6 = R_3\left(2 + \frac{2\sqrt{1-t}}{\sqrt{3}} + \cdots\right),\\
&x_3 = R_6\left(\frac{-3+\sqrt{t-1}}{3\sqrt[3]{2}} + \cdots\right), && x_4 = R_6\left(\frac{-3+\sqrt{t-1}}{3\omega\sqrt[3]{2}} + \cdots\right), && x_8 = R_6\left(\frac{-3+\sqrt{t-1}}{3\bar{\omega}\sqrt[3]{2}} + \cdots\right),\\
&x_9 = R_6\left(\frac{-3-\sqrt{t-1}}{3\sqrt[3]{2}} + \cdots\right), && x_5 = R_6\left(\frac{-3-\sqrt{t-1}}{3\bar{\omega}\sqrt[3]{2}} + \cdots\right), && x_7 = R_6\left(\frac{-3-\sqrt{t-1}}{3\omega\sqrt[3]{2}} + \cdots\right),
\end{aligned}$$

where the expansions of r and p, which are the roots of $r^3 - 3r^2 + 4t = 0$ and $4tp^6 + 4tp^3 + 1 = 0$, have been given to order $O(1-t)$. For the remainder of this section, let the groups G_i (for $i = 9$, 18, 72 and 216) be the groups that were defined in the proof of Lemma 4.2. Since there are an odd number of complex roots when (s,t) is close to $(0,1)$, in order to compute the Galois group of (7.4) over $\mathbb{Q}(s,t)$, we should introduce another group G_{432} that is generated by G_{216} and $(39)(45)(78)$, since this is the permutation that interchanges the complex roots. The labels of these groups in [7] are given in the following table. Of course, the generators we have given here differ from those given in [7].

group	notation of [7]	parity
G_9	$E(9)$	$+$
G_{18}	$S(3)[\frac{1}{2}]S(3) = E(9):2$	$+$
G_{72}	$E(9):Q_8$	$+$
G_{216}	$E(9):2A_4$	$+$
G_{432}	$E(9):2S_4$	$-$

Theorem 7.6. *Let F denote the splitting field of (7.4). We have $\mathrm{Gal}(F/\mathbb{C}(s,t)) \simeq G_{216}$ and $\mathrm{Gal}(F/\mathbb{Q}(s,t)) \simeq G_{432}$, and an explicit construction of the roots of (7.4) in terms of radicals may be given as follows. In each step we adjoin radicals according to the normal series*

$$G_{432} \trianglerighteq G_{216} \trianglerighteq G_{72} \trianglerighteq G_{18} \trianglerighteq G_9 \trianglerighteq 1.$$

$\mathbb{Z}_2 \simeq G_{432}/G_{216}$. *For the first step in solving (7.4), we fix a cube roots of unity ω and determine $\sqrt{-3} = \omega - \omega^2$. This step is omitted if the base field contains cube roots of unity.*

$\mathbb{Z}_3 \simeq G_{216}/G_{72}$. *Next, we fix a cube root of $2s^2(t^2-4s)$ and determine the elements u_1, u_2, u_3 by*

$$u_i = \frac{st}{2s + \omega^i \sqrt[3]{2s^2(t^2-4s)}}.$$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \simeq G_{72}/G_{18}$. *Next, we fix square roots of u_1, u_2, u_3 and determine elements y_1, y_2, y_3, y_4 by*

$$y_{\substack{1\\2}} = -\sqrt{u_1}\sqrt{u_2} \mp \sqrt{u_2}\sqrt{u_3} \pm \sqrt{u_3}\sqrt{u_1},$$
$$y_{\substack{3\\4}} = +\sqrt{u_1}\sqrt{u_2} \mp \sqrt{u_2}\sqrt{u_3} \mp \sqrt{u_3}\sqrt{u_1}.$$

$\mathbb{Z}_2 \simeq G_{18}/G_9$. *Next, we determine elements $z_1, z_{\bar{1}}, \ldots, z_4, z_{\bar{4}}$ determined by the four equalities E_1, E_2, E_3, E_4 and six equalities $E_{2,1,3,4}, E_{3,4,2,1}, E_{1,3,2,4}, E_{2,4,1,3}, E_{4,1,2,3}, E_{2,3,4,1}$, where E_a and $E_{a,b,c,d}$ denote the equalities*

$$E_a: \quad z_a + z_{\bar{a}} = 2 - 4t - 7y_a + s^{-1}y_a^3,$$
$$E_{a,b,c,d}: \quad (z_a - z_{\bar{a}})(z_b - z_{\bar{b}}) = (st)^{-1}(1+s-t)(y_a+y_b)\left(3s(y_c-y_d) + \sqrt{-3}y_c y_d(y_a-y_b)\right).$$

Once $z_1 - z_{\bar{1}}$ is determined, for example, by fixing the square root

$$z_1 - z_{\bar{1}} = \frac{\sqrt{z_1 - z_{\bar{1}} \cdot z_2 - z_{\bar{2}}}\sqrt{z_1 - z_{\bar{1}} \cdot z_3 - z_{\bar{3}}}}{\sqrt{z_2 - z_{\bar{2}} \cdot z_3 - z_{\bar{3}}}},$$

the elements $z_1, z_{\bar{1}}, \ldots, z_4, z_{\bar{4}}$ are then uniquely determined.

$\mathbb{Z}_3 \times \mathbb{Z}_3 \simeq G_9/1$. *Finally, we fix cube roots of the elements $z_1, z_{\bar{1}}, \ldots, z_4, z_{\bar{4}}$ subject to the constraint*

$$\sqrt[3]{z_1}\sqrt[3]{z_2}\sqrt[3]{z_3}\sqrt[3]{z_4}\sqrt[3]{z_{\bar{1}}}\sqrt[3]{z_{\bar{2}}}\sqrt[3]{z_{\bar{3}}}\sqrt[3]{z_{\bar{4}}} = 1 - 6s - 3s^2 + 4st. \tag{7.5}$$

A root x of the equation (7.4)*, is then given by*

$$
\begin{aligned}
3x = \quad & 1 && + \frac{\sqrt[3]{z_2}\,\sqrt[3]{z_4}\,\sqrt[3]{z_{\bar{3}}}}{\sqrt[3]{z_1}\,\sqrt[3]{z_{\bar{1}}}} + \frac{\sqrt[3]{z_3}\,\sqrt[3]{z_{\bar{2}}}\,\sqrt[3]{z_{\bar{4}}}}{\sqrt[3]{z_1}\,\sqrt[3]{z_{\bar{1}}}} \\
& + \frac{\sqrt[3]{z_{\bar{1}}}\,\sqrt[3]{z_{\bar{3}}}\,\sqrt[3]{z_{\bar{4}}}}{\sqrt[3]{z_2}\,\sqrt[3]{z_{\bar{2}}}} && + \frac{\sqrt[3]{z_2}\,\sqrt[3]{z_3}\,\sqrt[3]{z_{\bar{1}}}}{\sqrt[3]{z_4}\,\sqrt[3]{z_{\bar{4}}}} + \frac{\sqrt[3]{z_4}\,\sqrt[3]{z_{\bar{1}}}\,\sqrt[3]{z_{\bar{2}}}}{\sqrt[3]{z_3}\,\sqrt[3]{z_{\bar{3}}}} \\
& + \frac{\sqrt[3]{z_1}\,\sqrt[3]{z_3}\,\sqrt[3]{z_4}}{\sqrt[3]{z_2}\,\sqrt[3]{z_{\bar{2}}}} && + \frac{\sqrt[3]{z_1}\,\sqrt[3]{z_2}\,\sqrt[3]{z_{\bar{4}}}}{\sqrt[3]{z_3}\,\sqrt[3]{z_{\bar{3}}}} + \frac{\sqrt[3]{z_1}\,\sqrt[3]{z_{\bar{2}}}\,\sqrt[3]{z_{\bar{3}}}}{\sqrt[3]{z_4}\,\sqrt[3]{z_{\bar{4}}}}.
\end{aligned} \tag{7.6}
$$

Remark 7.7. *Ignoring the difficult issue of how to choose the $\sqrt[3]{R_{ij}}$ correctly, we may give the roots of* (7.4) *as*

$$3x = 1 + \sqrt[3]{R_{01}} + \sqrt[3]{R_{02}} + \sqrt[3]{R_{10}} + \sqrt[3]{R_{11}} + \sqrt[3]{R_{12}} + \sqrt[3]{R_{20}} + \sqrt[3]{R_{21}} + \sqrt[3]{R_{22}},$$

where the R_{ij} are the eight roots of $R^2 - (2 + 3y - s^{-1}y^3)R + (1 + y)^3 = 0$, where y is any of the roots of $y^4 - 6sy^2 - 4sty - 3s^2 = 0$ (or any of the y_i constructed above).

Proof. We first prove that $\mathrm{Gal}(F/\mathbb{C}(s,t)) = G_{216}$ and $\mathrm{Gal}(F/\mathbb{Q}(s,t)) = G_{432}$. Since $\Gamma(\sqrt{-3})$ acts on the nine roots by permuting them according to G_{216} (by Lemma 4.2), we clearly have $\mathrm{Gal}(F/\mathbb{C}(s,t)), \mathrm{Gal}(F/\mathbb{Q}(s,t)) \supseteq G_{216}$. Next, we have

$$
\begin{aligned}
& x_1x_4x_5 + x_3x_6x_5 + x_2x_8x_5 + x_7x_9x_5 + x_1x_2x_6 + x_1x_3x_7 \\
& + x_2x_4x_7 + x_3x_4x_8 + x_6x_7x_8 + x_2x_3x_9 + x_4x_6x_9 + x_1x_8x_9 \\
& = -4t,
\end{aligned}
$$

and the sabilizer in S_9 of the element on the left is exactly G_{432} (over $\mathbb{Q}(s,t)$ this element is distinct from all of its $839 = 9!/432 - 1$ conjugates in S_9). Since the element on the right is in $\mathbb{Q}(s,t)$, we must have

$$G_{432} \supseteq \mathrm{Gal}(F/\mathbb{C}(s,t)), \mathrm{Gal}(F/\mathbb{Q}(s,t)) \supseteq G_{216}.$$

Finally, the square root of the discriminant of (7.4) is

$$\prod_{i<j}(x_i - x_j) = 2^{12}3^4s^5(1 + s - t)^2(t^2 - 4s)^2\sqrt{-3},$$

which implies that $\mathrm{Gal}(F/\mathbb{C}(s,t)) = G_{216}$ and $\mathrm{Gal}(F/\mathbb{Q}(s,t)) = G_{432}$ since G_{216} consists entirely of even permutations and G_{432} does not.

For the first step in the solution by radicals, we note that the elements u_1, u_2, u_3 are permuted by G_{216} and are roots of a cubic over $\mathbb{C}(s,t)$. The three displayed solutions are ordered so that G_{216} acts by cycling the u_1, u_2, u_3 according to $(u_1u_2u_3)$, as required in Table (7.3). The solutions for y_1, y_2, y_3, y_4 follow by inverting the equations in Table (7.3). The effect of these first two steps is to ensure that the y_1, y_2, y_3, y_4 are ordered so that

$$\Delta := \prod_{i<j}(y_i - y_j) = 48s^2(t^2 - 4s)\sqrt{-3}.$$

The z_i will be found in two steps. First, for some $c_i \in \mathbb{C}(s,t)$, we have

$$\begin{aligned}
z_1 + z_{\bar{1}} &= c_0 + c_1 y_1 + c_2 y_1^2 + c_3 y_1^3,\\
z_2 + z_{\bar{2}} &= c_0 + c_1 y_2 + c_2 y_2^2 + c_3 y_2^3,\\
z_3 + z_{\bar{3}} &= c_0 + c_1 y_3 + c_2 y_3^2 + c_3 y_3^3,\\
z_4 + z_{\bar{4}} &= c_0 + c_1 y_4 + c_2 y_4^2 + c_3 y_4^3,
\end{aligned}$$

since the $z_i + z_{\bar{i}}$ and the y_i are permuted identically under G_{216}. The c_i may be found be observing that the Δc_i are polynomials in s and t; the results are given in the equalities E_a. By the same reasoning and setting $\delta_i = z_i - z_{\bar{i}}$, we have

$$\begin{aligned}
(\delta_1\delta_2 + \delta_3\delta_4)/(y_2y_3 - y_1y_4) &= a_0 + a_1\,(y_1y_3 + y_2y_4) + a_2\,(y_1y_3 + y_2y_4)^2,\\
(\delta_2\delta_3 + \delta_1\delta_4)/(y_2y_4 - y_1y_3) &= a_0 + a_1\,(y_1y_2 + y_3y_4) + a_2\,(y_1y_2 + y_3y_4)^2,\\
(\delta_1\delta_3 + \delta_2\delta_4)/(y_1y_2 - y_3y_4) &= a_0 + a_1\,(y_2y_3 + y_1y_4) + a_2\,(y_2y_3 + y_1y_4)^2,\\
(\delta_1\delta_3 - \delta_2\delta_4)/(y_2y_3 - y_1y_4) &= b_0 + b_1\,(y_1y_2 + y_3y_4) + b_2\,(y_1y_2 + y_3y_4)^2,\\
(\delta_1\delta_2 - \delta_3\delta_4)/(y_1y_3 - y_2y_4) &= b_0 + b_1\,(y_2y_3 + y_1y_4) + b_2\,(y_2y_3 + y_1y_4)^2,\\
(\delta_1\delta_4 - \delta_2\delta_3)/(y_1y_2 - y_3y_4) &= b_0 + b_1\,(y_1y_3 + y_2y_4) + b_2\,(y_1y_3 + y_2y_4)^2,
\end{aligned} \tag{7.7}$$

for some $a_i, b_i \in \mathbb{C}(s,t)$. The Vandermonde determinant on the right hand side of the equations for the a_i or b_i is precisely Δ. Since $y_1y_3 - y_2y_4 = 4u_1\sqrt{u_2}\sqrt{u_3}$ with similar equations for the other denominators in (7.7), we must have that $2u_1u_2u_3\Delta a_i = st\Delta a_i$ is a polynomial in s and t (likewise for the b_i). These elements can be calculated as

$$\begin{aligned}
a_0 &= 6t^{-1}(1+s-t), \quad a_1 = +\sqrt{-3}(st)^{-1}(1+s-t), \quad a_2 = 0\\
b_0 &= 6t^{-1}(1+s-t), \quad b_1 = -\sqrt{-3}(st)^{-1}(1+s-t), \quad b_2 = 0,
\end{aligned}$$

and the solutions for the various products $\delta_a\delta_b$ are given in the equalities $E_{a,b,c,d}$. Finally, the solutions for the roots follows by inverting the definitions of the $r_{i,j}$ in Table (7.2). There are 27 solutions for the $r_{i,j}$ from inverting the definitions of the z_i and $z_{\bar{i}}$. We begin by fixing cube roots $\sqrt[3]{z_1} = \sqrt[3]{r_{2,0}}\sqrt[3]{r_{2,1}}\sqrt[3]{r_{2,2}}$, with similar equations for the other z_i and $z_{\bar{i}}$. With these choices of the cube roots, we notice that (7.6) is immediately a correct solution for x_1, and that

$$\begin{aligned}
\sqrt[3]{z_1}\sqrt[3]{z_2}\sqrt[3]{z_3}\sqrt[3]{z_4}\sqrt[3]{z_{\bar{1}}}\sqrt[3]{z_{\bar{2}}}\sqrt[3]{z_{\bar{3}}}\sqrt[3]{z_{\bar{4}}} &= r_{0,1}r_{0,2}r_{1,0}r_{2,0}r_{1,2}r_{2,1}r_{1,1}r_{2,2}\\
&= (y_1+1)(y_2+1)(y_3+1)(y_4+1)\\
&= 1 - 6s - 3s^2 + 4st.
\end{aligned}$$

Without this constraint, the right hand side of (7.6) has 27 values if all possible cube roots are considered. Thus, when choosing arbitrary cube roots of the z_i and $z_{\bar{i}}$, this condition needs to be enforced so that the expression in (7.6) has only nine possible values. □

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